The Mathematical Theory of Wavelets

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ABSTRACT. We present an overview of some aspects of the mathematical theory of wavelets. These notes are addressed to an audience of mathematicians familiar with only the most basic elements of Fourier Analysis. The material discussed is quite broad and covers several topics involving wavelets. Though most of the larger and more involved proofs are not included, complete references to them are provided. We do, however, present complete proofs for results that are new (in particular, this applies to a recently obtained characterization of "all" wavelets in section 4).

1. Introduction

A *wavelet* is a function ψ in $L^2(\mathbb{R})$ such that the system

$$\psi_{jk}(x) \equiv 2^{j/2}\,\psi(2^j x - k) \tag{1.1}$$

j, k $\epsilon\,\mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$. Observe that if τ_k is the translation operator mapping ψ into $(\tau_k\,\psi)(x) = \psi(x-k)$, $k\,\epsilon\,Z$, and D^j is the dilation operator defined by $(D^j\,\psi)(x) = 2^{j/2}\,\psi(2^j x)$, then the system $\{\psi_{jk}\}$ is obtained by *first* applying the translation τ_k to ψ and, *secondly*, the dilation D^j to the function $\tau_k\psi$. As we shall see later on, it is important to respect this order of applying these operators: the translation operator is applied before the dilation operator.

Two examples of wavelets were known for a long time: the *Haar wavelet* and the *Shannon wavelet*. The former is the function

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < 1/2 \\ -1, & \text{if } 1/2 \leqslant x < 1 \\ 0, & \text{elsewhere} \end{cases} \quad . \tag{1.2}$$

The latter is the function ψ whose Fourier transform is

(1.3)
$$\widehat{\psi}\,(\xi) = \begin{cases} 1, & \text{if } \xi \,\epsilon\, (-1,-1/2] \cup [1/2,1) \\ 0, & \text{elsewhere} \end{cases}.$$

The Fourier transform we shall use is given by the equality

(1.4)
$$\hat{f}\,(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi\, i\, \xi\, x}\, dx$$

whenever $f \,\epsilon\, L^1(\mathbb{R})$. We assume that the reader is acquainted with the basic L^2- theory of the Fourier transform. In particular, $(\tau_k\, f)^\wedge(\xi) = e^{-2\pi\, ik\,\xi}\,\hat{f}(\xi)$ and $(D^j f)^\wedge\,(\xi) = 2^{-j/2}\,\hat{f}\,(2^{-j}\,\xi)$. Thus, translations by k are converted by the Fourier transform into *modulations* by $-k$ (multiplication by $e^{-2\pi\, ik\,\xi}$); the dilations D^j become the dilations D^{-j} after taking the Fourier transform. The Plancherel theorem, the fact that $\{2^j S\},\ j \,\epsilon\, \mathbb{Z}$, is a partition of $\mathbb{R} - \{0\}$ when $S = (-1,-1/2] \cup [1/2,1)$, and the completeness of the system $\{e^{2\pi\, ik\,\xi}\}$, $k \,\epsilon\, \mathbb{Z}$, in $L^2(S)$, immediately imply that the Shannon function ψ in (1.3) is a wavelet. That the Haar function defined in (1.2) is a wavelet has been well known since it was introduced in 1910 [Ha]. In any case, this is an easy application of the characterizations of wavelets we shall present.

In the early eighties many different constructions of wavelets were discovered. This included several other similar methods of reproducing functions. For example, pairs of systems $\{\phi_{jk}\}$ and $\{\psi_{jk}\}, j, k \,\epsilon\, \mathbb{Z}$, were introduced so that for any $f \,\epsilon\, L^2(\mathbb{R})$ we have the reproducing formula

(1.5)
$$f = \sum_{j,k\,\epsilon\,\mathbb{Z}} \langle f, \phi_{jk} \rangle\, \psi_{jk}$$

for all $f \,\epsilon\, L^2(\mathbb{R})$.

We will present a careful accounting of who produced the results we describe throughout the text as well as in an appendix at the end of this exposition. Soon after the “new wavelets” were introduced it became apparent that they had important applications in various different areas. This attracted many investigators whose principal interest was in these applications. Perhaps this detracted attention from the mathematical theory that is associated with wavelets and similar concepts. Our purpose is to present some of this theory. It is our belief that it is a beautiful subject connected to many areas of mathematics.

It is clear from the little that has been presented so far that the Fourier transform must play an important role in the study of bases and similar systems that are constructed by applying translations, dilations and modulations to a specific function. Let us illustrate this by presenting a characterization of those $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k}\}, j, k \in \mathbb{Z}$, is an orthonormal system:

PROPOSITION I. *Suppose* $\psi \in L^2(\mathbb{R})$. *Then* $\{\psi(. - k) : k \in \mathbb{Z}\}$ *is an orthonormal system if and only if*

$$\text{(A)} \qquad \sum_{k \in \mathbb{Z}} |\widehat{\psi}(\xi + k)|^2 = 1 \ \ \textit{for a.e. } \xi \in \mathbb{R}.$$

The proof of this fact is very simple. The orthonormality condition is $\langle \psi(. - j), \psi(. - l) \rangle = \delta_{jl}$, which, by the Plancherel theorem, is equivalent to

$$\delta_{jl} = \int_{\mathbb{R}} \widehat{\psi}(\xi) \overline{\widehat{\psi}(\xi)} e^{-2\pi i (j-l)\xi} \, d\xi.$$

We can then "periodize" this integral so that it takes the form

$$\int_0^1 \sum_{k \in \mathbb{Z}} |\widehat{\psi}(\xi + k)|^2 e^{-2\pi i (j-l)\xi} \, d\xi$$

and we see that the orthonormality condition is equivalent to the statement that the 1-periodic function $\sum_{k \in \mathbb{Z}} |\widehat{\psi}(\xi + k)|^2$, which clearly belongs to $L^2([0, 1))$, has Fourier coefficients 0 corresponding to all non-zero frequencies and the zero-coefficient is 1. But this is equality (A).

PROPOSITION II. *The systems* $\{\psi_{j_1,k}\}$ *and* $\{\psi_{j_2,k}\}, k \in \mathbb{Z}$, *are orthogonal to each other whenever* $j_1 \neq j_2$ *if and only if*

$$\text{(B)} \qquad \sum_{k \in \mathbb{Z}} \widehat{\psi}(\xi + k) \overline{\widehat{\psi}(2^j(\xi + k))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R} \textit{ whenever } j \geqslant 1.$$

By a change of variable the orthogonality condition can be reduced to the case $j_1 = 0$ and $j_2 \geqslant 1$. A periodization argument, just like the one we just described then gives us equality (B).

Thus, we see that the characterization of *all* wavelets is reduced to finding a condition that implies the completeness of the system $\{\psi_{jk}\}, j, k \in \mathbb{Z}$. It turns out that, again, a simple equality, involving the Fourier transform of ψ, provides us with such a characterization of completeness:

PROPOSITION III. *(The characterization of all wavelets in* $L^2(\mathbb{R})$*). A function* $\psi \in L^2(\mathbb{R})$ *is a wavelet if and only if the system* $\{\psi_{jk}\}$ *is orthonormal and*

(C) $$\sum_{j \epsilon \mathbb{Z}} |\,\widehat{\psi}\,(2^j\,\xi)\,|^{\,2} = 1 \quad \textit{for a.e } \xi\,\epsilon\,\mathbb{R}.$$

Unlike Proposition I, this result is not immediate. It is also quite new. We shall discuss its proof in the sequel. For the moment, let us make some observations.

The characterization of orthonormality involved averaging (summing) over the group of integral translations. Since the group of dyadic dilations also plays a basic role in the definition of a wavelet it is natural to expect that averaging over this last group plays a part in this characterization. In fact, this is precisely what is the case in equality (C). What is surprising, however, is that there is a characterization of all wavelets that involves only sums over dilations:

PROPOSITION IV. *(Another characterization of all wavelets) Suppose* $\psi\,\epsilon\,L^2(\mathbb{R})$*, then* ψ *is a wavelet if and only if* $||\psi||_2 \geqslant 1$*, equality* (C) *is satisfied, and*

(D) $$t_q\,(\xi) = \sum_{j \geqslant 0} \widehat{\psi}(2^j\,\xi)\,\overline{\widehat{\psi}(2^j(\xi+q))} = 0 \textit{ for a.e. } \xi\,\epsilon\,\mathbb{R},$$

whenever q is an odd integer.

Let us explain the role played by the hypothesis $||\psi||_2 \geqslant 1$. Let H be a separable Hilbert space and $\mathcal{E} = \{e_\alpha \,:\, \alpha\,\epsilon\,\mathcal{A}\}$ a countable collection of vectors in H ($\mathcal{A}$ can be **N** or $\{(j,k) : j,k\epsilon\,\mathbb{Z}\}$) such that $||u||^2 = \sum_{\alpha\,\epsilon\,\mathcal{A}} |(u,e_\alpha)|^2$ for each $u\,\epsilon\,H$. Such a collection $\mathcal{E}$ is then called a *tight frame* (of constant 1) for H. If $||e_\alpha|| \geqslant 1$ for all $\alpha\,\epsilon\,\mathcal{A}$ then, letting $u = e_{\alpha_0}$, we have

$$||e_{\alpha_0}||^2 \;=\; \sum_{\alpha\,\epsilon\,\mathcal{A}} |(e_{\alpha_0},\, e_\alpha)|^2 \;=\; ||e_{\alpha_0}||^4 \,+ \sum_{\alpha \neq \alpha_0} |(e_{\alpha_0},\, e_\alpha)|^2.$$

Hence,

$$||e_{\alpha_0}||^2\,(1 - ||e_{\alpha_0}||^2) \;=\; \sum_{\alpha\,\neq\,\alpha_0} |(e_{\alpha_0},\, e_\alpha)|^2.$$

Because of our assumption that $||e_{\alpha_0}|| \geqslant 1$, the left side of this equality cannot be strictly bigger than 0, while the right side cannot be negative. It follows that $||e_{\alpha_0}|| = 1$ and $(e_{\alpha_0}, e_\alpha) = 0$ for all α_0 and $\alpha \neq \alpha_0$. That is, $\mathcal{E}$ is an orthonormal basis (see pages 336-7 of [HW] for a more complete account of these matters).

The two equalities (C) and (D) characterize those $\psi\,\epsilon\,L^2\,(\mathbb{R})$ for which the system $\{\psi_{jk}\}$, j,k in $\mathbb{Z}$, is a tight frame of constant 1 for $L^2(\mathbb{R})$. The condition $||\psi||_2 \geqslant 1$ assures us that this system is an orthonormal basis; that is, that ψ is a wavelet.

The four equations (A), (B), (C) and (D) not only provide us with a rather simple characterization of all wavelets, but they are most useful for constructing large classes of wavelets.

For example, it follows immediately from (A) or (C) that if ψ is a wavelet then $|\widehat{\psi}(\xi)| \leqslant 1$ a.e. Since $||\psi||_2 = ||\widehat{\psi}||_2 = 1$ this means that $\{\xi : \widehat{\psi}(\xi) \neq 0\}$ must have measure at least 1. The Shannon wavelet is an example for which this set has measure precisely 1. It is natural to consider the class of all wavelets ψ such that $\mathbf{W} = \mathbf{W}_\psi = \{\xi : \widehat{\psi}(\xi) \neq 0\}$ has measure 1. For such ψ we clearly must have $|\widehat{\psi}| = \chi_W$. It is natural to call the class of such wavelets the collection of *Minimally Supported Frequency* (MSF) wavelets. It is an easy exercise to show that the MSF wavelets are characterized as the class of all $\psi \in L^2(\mathbb{R})$ such that $|\widehat{\psi}(\xi)|$ assumes only the values 0 or 1 a.e. and equations (A) and (C) are satisfied. The sets $\mathbf{W} = \mathbf{W}_\psi$ on which the Fourier transform of MSF wavelets is not zero are called *wavelet sets*. (A) and (C) are equivalent to the statement:

THEOREM (1.1). *$\boldsymbol{W}$ is a wavelet set if and only if each of the collections $\{\boldsymbol{W} - k\}, k$ in $\mathbb{Z}$, and $\{2^j\boldsymbol{W}\}, j$ in $\mathbb{Z}$ is a partition of $\mathbb{R}$.*

In the course of this exposition the reader will find many examples of wavelets constructed by the use of these 4 equations and in the appendix we will give a still larger class of wavelets obtained by these means.

We shall also consider the subject of wavelets $\psi \in L^2(\mathbb{R}^n)$. Not only will we show many of the various properties they enjoy, but we will generalize the concept by showing how other dilations and translations can be used for obtaining orthonormal bases or tight frames from a particular function (or a collection of functions); moreover, we will extend all these matters to higher dimensions. In order to do this most efficiently it is useful to discuss "continuous wavelets" associated with $\mathbb{R}^n$. For many considerations the theory of these wavelets is simpler.

2. Continuous Wavelets in One and More Dimensions

Let G be the *affine group* associated with $\mathbb{R}$ consisting of all

$$(a, b) \in \mathbb{R} \times \mathbb{R},\ a \neq 0, \quad \text{with the group operation}$$

$$(c, d) \circ (a, b) = (ac, b + \frac{d}{a}).$$

This operation is consistent with the action of $g = (a, b) \in G$ on x in $\mathbb{R}$ given by $g(x) = a(x + b)$. Observe that $g^{-1} = (a, b)^{-1} = (a^{-1}, -ab)$. For $\psi \in L^2(\mathbb{R})$ let

$$(T_g\,\psi)(x) = \frac{1}{\sqrt{|a|}}\,\psi\,(\frac{x}{a} - b) = \frac{1}{\sqrt{|a|}}\,\psi(g^{-1}(x)) \equiv \psi_{a,b}(x). \tag{2.1}$$

Then $g \longrightarrow T_g$ is a unitary representation of G acting on $L^2(\mathbb{R})$.

The mapping W_ψ taking $f \in L^2(\mathbb{R})$ into the function

$$(W_\psi f)(g) = \int_{\mathbb{R}} f(x)\,\overline{(T_g\,\psi)(x)}\,dx \;=\; \langle f,\,\psi_{a,b}\rangle$$

on G is the (*continuous*) *wavelet transform of f*. A goal in wavelet theory is to find a condition on ψ that allows us to reconstruct f from its wavelet transform via the reproduction formula

$$f(x) \;=\; \int_G \langle f,\,\psi_g\rangle\,\psi_g\,(x) d\lambda\,(g) \;=\; \int_G (W_\psi\, f)(g)\,T_g\,\psi\,(x)\,d\lambda\,(g), \tag{2.2}$$

where λ is (left) Haar measure on $G\,(d\lambda(a,b) \;=\; \frac{da\,db}{|a|})$. One can consider (1.5) as a discrete version of this reproduction formula when $\varphi = \psi$.

This condition, the *admissibility condition* for ψ, was discovered by Calderón in 1964 [C] and can be expressed in the form

$$1 = \int_{\mathbb{R}-\{0\}} |\widehat{\psi}(a\,\xi)|^2\,\frac{da}{|a|} \tag{2.3}$$

for a.e. ξ. We will show the equivalence of (2.2) and (2.3) in a considerably more general context.

It is clear that (2.3) is the "continuous" analog of equality (C). In fact it is much more than an analog. Suppose that $\psi\,\epsilon\,L^2\,(\mathbb{R})$ satisfies (C), then

$$\log 2 = \int_1^2 \frac{da}{a} \;= \int_1^2 \sum_{j\,\epsilon\,\mathbb{Z}} |\widehat{\psi}(2^j a)|^2\,\frac{da}{a} = \sum_{j\,\epsilon\,\mathbb{Z}} \int_{2^j}^{2^j+1} |\widehat{\psi}(a)|^2\,\frac{da}{a}$$

$$= \int_0^\infty |\widehat{\psi}(a)|^2\,\frac{da}{a} \;=\; \int_0^\infty |\widehat{\psi}(a\,\xi)|^2\,\frac{da}{a}.$$

for $\xi > 0$. Thus, it follows that, after a renormalization, ψ satisfies (2.3).

This shows, essentially, that each wavelet is also a continuous wavelet. On the other hand, it is clear that the converse is not true; being a wavelet is more restrictive than being a continuous wavelet.

Let us also observe that, in the continuous case, the order of the operation of translation (by $-b$) followed by dilation (by $\frac{1}{a}$), as performed in the definition of $\psi_{a,b}$, can be reversed and we would, again, have that the same admissibility condition (2.3) is equivalent to the reproducing formula (2.2). More precisely, if G is endowed with the operation $(a,b)\cdot(c,d) = (ac,\; ad+b)$ (that corresponds to the action $x \longrightarrow ax+b$ on $\mathbb{R}$) and

$$(S_g\,\psi)(x) = \frac{1}{\sqrt{|a|}}\,\psi\,(g^{-1}x) = \frac{1}{\sqrt{|a|}}\,\psi\,(\frac{x-b}{a}) \equiv \widetilde{\psi}_{a,b}\,(x),$$

we have

$$\int_G \langle f,\, \psi_{a,b}\rangle\, \psi_{a,b}\,(x)\, \frac{dadb}{|a|} = \int_{\widetilde{G}} \langle f,\, \widetilde{\psi}_{a,b}\rangle\, \widetilde{\psi}_{a,b}\,(x)\, \frac{dadb}{a^2} \tag{2.4}$$

for all $f \,\epsilon\, L^2(\mathbb{R})$ ($\widetilde{G}$ is the "new" version of the affine group with this last multiplication, so that its left Haar measure is $\frac{dadb}{a^2}$).

Let us now pass to the extensions of these notions and results to n dimensions. The *Full Affine Group of Motions on* $\mathbb{R}^n$, $G^{\#}$, consists of all pairs $(a,b)\,\epsilon\, GL(n,\mathbb{R}) \times \mathbb{R}^n$ (endowed with the product topology) together with the operation

$$(\alpha,\beta)\cdot(a,b) = (\alpha a, b + a^{-1}\,\beta).$$

This operation is associated with the action $x \longrightarrow a(x+b)$ on $\mathbb{R}^n$. The subgroup

$$\mathcal{N} = \{(a,b)\,\epsilon\, G^{\#} : a = I,\ b\,\epsilon\,\mathbb{R}^n\}$$

is clearly a normal subgroup of $G^{\#}$.

We consider a class of subgroups, $\{G\}$, of $G^{\#}$ of the form

$$G = \{(a,b)\,\epsilon\, G^{\#} : a\,\epsilon\, D, b\,\epsilon\,\mathbb{R}^n\},$$

where D is a closed subgroup of $GL(n,\mathbb{R})$. We can identify D with the subgroup $\{(a,b)\,\epsilon\, G : a\,\epsilon\, D, b = 0\}$) of G. We refer to D as the *dilation subgroup* and $\mathcal{N}$ will be called the *translation subgroup of* G.

If μ is left Haar measure for D, then $d\lambda(a,b) = d\mu(a)db$ is the element of left Haar measure for G.

Let T be the unitary representation of G acting on $L^2\,(\mathbb{R}^n)$ defined by

$$(T_{(a,b)}\,\psi)(x) = |\det a|^{-1/2}\,\psi(a^{-1}x - b) \equiv \psi_{a,b}(x) \tag{2.5}$$

for $(a,b)\,\epsilon\, G$ and $\psi\,\epsilon\, L^2\,(\mathbb{R}^n)$. Observe that $(a,b)^{-1} = (a^{-1}, -ab)$. We then have

$$(T_{(a,b)}\,\psi)^{\wedge}(\xi) = |\det a|^{\frac{1}{2}}\,\widehat{\psi}\,(a^*\,\xi)e^{-2\,\pi\, i\,\xi\,\cdot ab} \tag{2.6}$$

where a^* is the transpose of a.

The *wavelet transform* W_ψ *associated with* ψ is now defined by

$$(W_\psi f)(a,b) = \langle f,\, \psi_{a,b}\rangle = \int_{\mathbb{R}^n} f(y)\,\overline{\psi\,(a^{-1}y - b)}\,\frac{dy}{\sqrt{|\det a|}}$$

whenever $f\,\epsilon\, L^2(\mathbb{R}^n)$ and $(a,b)\,\epsilon\, G$. Our first goal is to find an admissibility condition for ψ that guarantees the general version of the Calderón reproducing formula

(2.7)
$$f(x) = \int_G \langle f,\, \psi_{a,b} \rangle \, \psi_{a,b}\,(x) d\,\lambda\,(a,b)$$

for all $f \,\epsilon\, L^2\,(\mathbb{R}^n)$. The analog of (2.4), involving the operation $(\alpha,\beta)\circ(a,b) = (\alpha a, \alpha b+\beta)$, is valid in this general case; hence, the same admissibility condition applies to both "versions" of G. This condition is

THEOREM (2.1). *Equality(2.7) is valid for all* $f \epsilon L^2(\mathbb{R}^n)$ *if and only if for a.e.* $\xi \neq 0$

(2.8)
$$\Delta_\psi\,(\xi) = \int_D |\widehat{\psi}(a^*\,\xi)\,|^2\, d\,\mu\,(a) = 1.$$

(compare with (2.3)).

The following argument also provides a (weak) meaning for (2.7).

W_ψ obviously maps $L^2(\mathbb{R}^n)$ into $L^\infty(G)$. We claim that, if (2.8) is satisfied, then W_ψ is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(G,\lambda)$:

$$||W_\psi\, f||^2_{L^2\,(G,\,\lambda)} = \int_D \int_{\mathbb{R}^n} |\langle f, \psi_{a,b}\rangle|^2 db d\mu\,(a) =$$

$$\int_D \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} \widehat{f}\,(\xi)\,\overline{\widehat{\psi}(a^*\,\xi)} e^{i\,2\,\pi\,\xi\cdot ab} d\,\xi|^2\,|\det a|\,db d\mu\,(a)$$

$$= \int_D [\int_{\mathbb{R}^n} |\{\widehat{f}\,\overline{\widehat{\psi}(a^*\cdot)}\}^\vee(ab)|^2\,|\det a|db]d\mu\,(a)$$

$$= \int_D [\int_{\mathbb{R}^n} |\{\widehat{f}\overline{\widehat{\psi}(a^*\cdot)}\}^\vee(b)|^2\,db]\,d\mu\,(a)$$

$$= \int_D \int_{\mathbb{R}^n} |\widehat{f}\,(\xi)|^2\,|\widehat{\psi}(a^*\,\xi)|^2\,d\,\xi\,d\,\mu\,(a) = \int_{\mathbb{R}^n} |\widehat{f}\,(\xi)|^2\,\Delta_\psi\,(\xi)\,d\,\xi$$

$$= ||\widehat{f}||_2^2 = ||f||_2^2.$$

By polarization, therefore, we have

(2.9)
$$\langle W_\psi\, f, W_\psi h\rangle_{L^2(G)} = \langle f, h\rangle_{L^2(\mathbb{R}^n)}$$

for all f and $g \,\epsilon\, L^2(\mathbb{R}^n)$. In particular, the adjoint, W_ψ^*, of W_ψ is a left inverse of W_ψ : $W_\psi^*\, W_\psi = I$. We also have shown that the reproducing formula is valid in the weak sense. We refer the reader to [S] for more general versions of this reproducing formula.

Suppose, on the other hand, W_ψ satisfies (2.9) (so that (2.7) is valid in the weak sense) and ξ_0 is a point of differentiability for the integral of Δ_ψ. Let $|\widehat{f}\,(\xi)|^2 = |B_r(\xi_0)|^{-1}\, X_{B_r(\xi_0)}\,(\xi)$, where $B_r\,(\xi_0)$ is the ball of radius $r > 0$ centered at ξ_0. Reversing the equality chain used to obtain (2.9) we see that

$$\frac{1}{|B_r\,(\xi_0)|}\int_{B_r\,(\xi_0)}\Delta_\psi\,(\xi)\,d\,\xi = 1$$

for all $r > 0$. Letting $r \longrightarrow 0$ we obtain $\Delta_\psi\,(\xi_0) = 1$. Since a.e. $\xi\,\epsilon\,\mathbb{R}^n$ is such a point of differentiability, the admissibility condition (2.8) is true and the theorem is established.

It is natural to ask: for what dilation groups D does there exist a $\psi\,\epsilon\,L^2(\mathbb{R}^n)$ satisfying the admissibility condition (2.8)? We shall call such groups *admissible*. When $n = 1$ and D is the group of non-zero numbers (or positive numbers) with multiplication being the group operation, it is clear that the admissibility condition is verified by any ψ such that $\widehat{\psi}$ is a bounded function supported in a compact set in $\mathbb{R} - \{0\}$ (appropriately scaled). Thus, there exists $\psi\,\epsilon\,L^2(\mathbb{R}^n)$ satisfying the admissibility condition . For the same reason, in $\mathbb{R}^n$, the group $D = \{aI : a\,\epsilon\,\mathbb{R} - \{0\}\}$ is also admissible. The group $SO(2)$, acting on $\mathbb{R}^2$, however, is not admissible. For if this group were admissible, then there exists $\psi\,\epsilon\,L^2(\mathbb{R}^2)$ such that

$$1 = \int_0^{2\,\pi} |\widehat{\psi}\,(e^{i\,\theta}\,\rho\,e^{i\,\varphi})|^2\,d\,\theta = \int_0^{2\,\pi} |\widehat{\psi}\,(\rho\,e^{i\,\theta})|^2\,d\,\theta.$$

Thus,

$$\int_0^\infty \rho\,d\rho = \int_0^\infty \rho\int_0^{2\,\pi} |\widehat{\psi}(\rho\,e^{i\,\theta})|^2\,d\,\theta\,d\,\rho = ||\widehat{\psi}||^2_{L^2(\mathbb{R}^2)} < \infty$$

which is clearly impossible.

However, the 1-parameter groups

$$D = \left\{a = e^{tL}\ : t\,\epsilon\,\mathbb{R},\ L = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right\} \tag{2.10}$$

and

$$D = \left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \neq 0, x, y\,\epsilon\mathbb{R}\right\} \tag{2.11}$$

are admissible.

The following (almost) characterization of the admissible groups D can be applied to see the validity of our claim about the last two examples.

THEOREM (2.2). *Let D be a closed subgroup of $DL(n,\mathbb{R})$ and consider the right action $\xi \longrightarrow a^*\,\xi$ of G on $\mathbb{R}^n$ for $\xi\,\epsilon\,\mathbb{R}^n$. Let*

$$D_\xi^\epsilon = \{a\,\epsilon\,D : ||a^*\,\xi - \xi|| \leqslant \epsilon\}$$

be the ϵ-stabilizer of ξ, for $\epsilon \geqslant 0$, and let $D_\xi = D_\xi^0$ be the stabilizer of ξ. If either

i. $|\det a| = \Delta(a)$ *for all* $a \,\epsilon\, D$, *or*
ii. $\{\xi \in \mathbb{R}^n : D_\xi$ *is noncompact*$\}$ *has positive Lebesgue measure*

holds, then D *is not admissible. If both i and*

iii. $\{\xi \epsilon \mathbb{R}^n : D^\epsilon_\xi$ *is non-compact for all* $\epsilon > 0\}$ *has positive Lebesgue measure*

fail, then D *is admissible.*

(Δ is the modular function of D : the Radon-Nikodym derivative $d\,\mu_l/d\,\mu_r$, where μ_l is left Haar measure and μ_r is right Haar measure on D, normalized so that $\Delta(I) = 1$, where I is the identity element of D.) The proof is by no means immediate and will appear in [LWWW].

When D is given by (2.11) and $\xi = (\xi_1, \xi_2)$ with $\xi_1 \neq 0$ we have

$$D^\epsilon_\xi = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : (x-1)^2 + y^2 \leqslant \epsilon^2/\xi_1^2 \right\};$$

This is compact for any positive ϵ, so condition 2 fails. Furthermore, for $a = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, $d\,\mu_l(a) = dxdy/x^2$ and $d\,\mu_r(a) = dxdy/|x|$ so $\Delta(a) = d\,\mu_l(a)/d\,\mu_r(a) = 1/|\det a|$ and so condition i is also invalid. Hence D is admissible. When D is a general 1-parameter subgroup of $GL(n;\mathbb{R})$ (i.e., $D = \{e^{tL} : t \,\epsilon\, \mathbb{R}\}$ for some $n \times n$ matrix L), D is unimodular and $\det(e^{tL}) = e^{t\,tr(L)}$ so i holds $\Leftrightarrow$ $tr(L) = 0$. In this case D is not admissible. When $tr(L) \neq 0$, i fails and it is easy to check that 2 also fails, so D is admissible.

A *homogeneous Galilei group* is a subgroup D of $GL(n+1, \mathbb{R})$ which is of the form (2.11) with y replaced by an $n \times 1$ column vector (i.e., a member of $\mathbb{R}^n$) and x replaced by an invertible $n \times n$ matrix satisfying various stipulations; e.g., x is an orthogonal matrix or the product of an orthogonal matrix and a non-zero scalar. The subgroup G of the affine group on $\mathbb{R}^{n+1}$ whose dilation group is D and whose translation group includes all $\mathbb{R}^{n+1}$ translations is then an *inhomogeneous Galilei group*.

Using Theorem (2.2), it follows easily that D is not admissible if we allow x to be orthogonal while D is admissible if we allow x to be the product of an orthogonal matrix and a non-zero scalar. In the second case, the family $\{T_g\,\psi : g \,\epsilon\, G\}$ determined by any continuous wavelet ψ and the reproducing formula associated with this family provide examples of what are known in physics as *coherent states*. The admissibility conditions for certain classes of these groups have been obtained by several authors. In general these derivations are rather complicated (see [Co]); theorem (2.2) does provide a simpler and more unified method for solving these problems.

As an introduction to the notion of "discretizing" continuous wavelets let us observe that the proof of Theorem (2.1) in the special case $n = 1$ and D the group $\{2^j : j \,\epsilon\, \mathbb{Z}\}$ shows that (C) in Proposition III is equivalent to the reproducing formula

(2.12)
$$f = \sum_{j\,\epsilon\,\mathbb{Z}} \int_{\mathbb{R}} \langle f, \psi_{jt} \rangle\, \psi_{jt}\, dt,$$

where $\psi_{jt}(x) \equiv 2^{j/2}\,\psi(2^j x - t)$ for $j\,\epsilon\,\mathbb{Z}$ and $t\,\epsilon\,\mathbb{R}$. That is, (2.12) is a "discretization" of (2.2) with the sum $\sum_{j\,\epsilon\,\mathbb{Z}}$ replacing the integral $\int_{\mathbb{R}} \cdot \frac{da}{|a|}$. The replacement of the integral over $\mathbb{R}$ in (2.12) by the sum $\sum_{k\,\epsilon\,\mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk}$ gives us the orthonormal wavelet expansion of f if ψ is such a wavelet. It turns out that there are other discretizations of (2.12). For example, if n is an odd integer and ψ is an o.n. wavelet, then

$$f = \sum_{j\,\epsilon\,\mathbb{Z}} \sum_{k\,\epsilon\,\mathbb{Z}} \frac{1}{n} \langle f, \psi_{j\frac{k}{n}} \rangle \psi_{j\frac{k}{n}}$$

with convergence in $L^2(\mathbb{R})$ (see [CS]). This is an example of a phenomenon known as *oversampling*. Letting n tend to ∞ we obtain (at least formally) the equality (2.12) which can be thought of as "the ultimate oversampling property" of a ψ satisfying (C). There are several examples of discretizations of the continuous wavelet properties. In order to appreciate the complexity of "discrete" over "continuous" wavelets we now present generalizations to non-dyadic wavelets, wavelet systems, and related families in n-dimensions.

Let $\Gamma = P\mathbb{Z}^n$ be a lattice in $\mathbb{R}^n$ (P any invertible $n \times n$ matrix) and A an $n \times n$ *dilation* matrix (each eigenvalue λ of A satisfies $|\lambda| > 1$) for which $A\,\Gamma \subset \Gamma$. For $\psi\,\epsilon\,L^2(\mathbb{R}^n)$ let $\mathcal{E} = \{\psi_{j,\gamma} : j\,\epsilon\,\mathbb{Z}, \gamma\,\epsilon\,\Gamma\}$, where

(2.13)
$$\psi_{j,\gamma}(x) = |\det A|^{j/2}\,\psi(A^j x - \gamma).$$

$\mathcal{E}$ is the *affine system* generated by ψ, the lattice Γ, and the dilation matrix A. ψ is a *wavelet relative to* Γ *and* A if and only if $\mathcal{E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. We shall also use the further notations: $\Gamma^* = \{\gamma'\,\epsilon\,\mathbb{R}^n : \langle \gamma', \gamma \rangle\,\epsilon\,\mathbb{Z}$ for all $\gamma\,\epsilon\,\Gamma\}$ and $B = A^*$, the transpose of A. Then $B\,\Gamma^* \subset \Gamma^*$. Let $\mathcal{S}$ be the set difference $\Gamma^* \setminus B\,\Gamma^*$. We then have the following generalizations of Proposition III and Proposition IV.

PROPOSITION III$'$. *ψ is a wavelet relative to Γ and A if and only if its affine system $\mathcal{E}$ is an orthonormal set in $L^2(\mathbb{R}^n)$ and*

(C')
$$\sum_{j\,\epsilon\,\mathbb{Z}} |\widehat{\psi}(B^j\,\xi)|^2 = |\det\,P| \quad \text{for a.e. } \xi\,\epsilon\,\mathbb{R}^n.$$

PROPOSITION IV$'$. *The affine system $\mathcal{E}$ generated by ψ is a tight frame of constant 1 if and only if equality (C') is satisfied and, for each $s\,\epsilon\,\mathcal{S}$,*

$$(D') \qquad \sum_{j \geqslant 0} \widehat{\psi}\,(B^j\,\xi)\,\overline{\widehat{\psi}\,(B^j(\xi+s))} = 0 \text{ for a.e. } \xi\,\epsilon\,\mathbb{R}^n.$$

ψ *is a wavelet if and only if* (C') *and* (D') *are satisfied with* $||\psi||_2 \geqslant 1$.

Now suppose A is of the form e^L for some $n \times n$ real matrix L; this is a very mild extra condition on A, e.g., it is automatically satisfied if A has no negative eigenvalues. Write A^t for e^{tL} and B^t for $(A^t)^* = e^{tL^*}$. Then $D = \{A^t : t\,\epsilon\,\mathbb{R}\}$ is a one parameter subgroup of $GL(n,\mathbb{R})$ with Haar measure $d\,\mu\,(A^t) = dt$. If ψ is a wavelet relative to $\Gamma = \mathbb{Z}^n$ and A, then equation (C') implies

$$\int_D |\widehat{\psi}\,((A^t)^*\,\xi)|^2\,d\,\mu\,(A^t) = \int_{-\infty}^{\infty} |\widehat{\psi}\,(B^t\,\xi)|^2\,dt$$

$$= \int_0^1 \sum_{j\,\epsilon\,\mathbb{Z}} |\widehat{\psi}\,(B^t(B^j\,\xi))|^2\,dt = 1$$

for *a.e.* $\xi\,\epsilon\,\mathbb{R}^n$. So ψ is a continuous wavelet relative to D. It is easy to construct examples where A belongs to larger subgroups (non 1-parameter) (D') and ψ remains a continuous wavelet relative to (D'). In view of Proposition IV$'$, we again conclude that it is relatively easy for a function $\psi\,\epsilon\,L^2(\mathbb{R}^n)$ to be a continuous wavelet but that far more structure is required for ψ to be a discrete wavelet. We are also led to pose the question of determining, for a given admissible group D $\subset GL(n,\mathbb{R})$, which discrete subgroups of D and which $\mathbb{R}^n$ lattices give rise to discrete systems analogous to $\mathcal{E}$ which are either orthonormal bases or tight frames for $L^2(\mathbb{R}^n)$.

Another observation is in order. In many situations it is appropriate to generate a wavelet basis with more than one funtion ψ. In the fourth section, for example, we shall see that $L = 2^n - 1$ functions $\psi^1, \ldots, \psi^L$ are needed to obtain MRA wavelets in n-dimensions. This is reflected in what follows.

With Γ and A as above and with L an integer $\geqslant 1$, we can associate with each family $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ an affine system $\mathcal{E}_\Psi = \{\psi^l_{j\,\gamma} : j\,\epsilon\,\mathbb{Z}, \gamma\,\epsilon\,\Gamma, 1 \leqslant l \leqslant L\}$ where $\psi^l_{j\,\gamma}$ is defined for ψ^l by (2.13). In addition, Ψ generates a *quasi-affine system* $\widetilde{\xi}_\Psi = \{\widetilde{\psi}^l_{j\,\gamma} : j\,\epsilon\mathbb{Z}, \gamma\,\epsilon\,\Gamma, 1 \leqslant l \leqslant L\}$ where

$$\widetilde{\psi}^l_{j\,\gamma}\,(x) = \left\{ \begin{array}{ll} \psi^l_{j\,\gamma}\,(x) & \text{if } j \geqslant 0 \\ |\det A|^j\,\psi^l\,(A^j(x-\gamma)) & \text{if } j < 0 \end{array} \right\}$$

Recall that an arbitrary collection $\{e_\alpha : \alpha\,\epsilon\,\mathcal{A}\}$ in $L^2(\mathbb{R}^n)$ is a *Bessel family* if there is a constant $C > 0$ such that

$$\sum_{\alpha\,\epsilon\,\mathcal{A}} |\langle f, e_\alpha\rangle|^2 \leqslant C\,||f||_2 \;\text{ for all }\; f\,\epsilon\,L^2(\mathbb{R}^n).$$

Suppose $\Psi = \{\psi^1, \ldots, \psi^L\}$ and $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^L\}$ are two families in $L^2(\mathbb{R}^n)$ for which the affine systems $\mathcal{E}_\Psi$ and $\mathcal{E}_\Phi$ are Bessel families. Then Φ is an *affine dual* of Ψ if $\mathcal{E}_\Phi$ is dual to $\mathcal{E}_\Psi$ in the sense that for all $f, g\,\epsilon\,L^2(\mathbb{R}^n)$,

$$\langle f, g\rangle = \sum_{l=1}^{L} \sum_{j\,\epsilon\mathbb{Z}} \sum_{\gamma\,\epsilon\,\Gamma} \langle f, \psi^l_{j\,\gamma}\rangle\, \langle \varphi^l_{j\,\gamma}, g\rangle. \tag{2.14}$$

Φ is a *quasi-affine dual* of Φ if (2.14) holds when $\psi^l_{j\,\gamma}$ and $\varphi^l_{j\,\gamma}$ are replaced by $\widetilde{\psi}^l_{j\,\gamma}$ and $\widetilde{\varphi}^l_{j\,\gamma}$.

THEOREM (2.3). *Using the above notation, suppose $\mathcal{E}_\Psi$ and $\mathcal{E}_\Phi$ are Bessel families. Then Φ is an affine dual of Ψ if and only if*

$$\sum_{l=1}^{L} \sum_{j\epsilon\mathbb{Z}} \widehat{\psi}^l\,(B^j\,\xi)\,\overline{\widehat{\varphi}^l\,(B^j\,\xi)} = |detP| \tag{2.15}$$

for a.e. $\xi\,\epsilon\,\mathbb{R}^n$ and, for each $s\,\epsilon\,\mathcal{S} = \Gamma^* \setminus B\,\Gamma^*$,

$$t_s(\xi) = \sum_{l=1}^{L} \sum_{j\geqslant 0} \widehat{\psi}^l\,(B^j\,\xi)\,\overline{\widehat{\varphi}^l\,(B^j(\xi+s))} = 0 \tag{2.16}$$

for a.e. $\xi\,\epsilon\,\mathbb{R}^N$.
Moreover, these two equations also characterize the relation that Φ is a *quasi-affine dual* of Ψ.

Note that (2.15) and (2.16) reduce to (C') and (D') when $L = 1$ and $\varphi^1 = \psi = \psi^1$. $\Psi = \{\psi^1, .., \psi^L\}$ is a *wavelet system* (relative to Γ and A) if $\mathcal{E}_\Psi$ is an *orthonormal basis* of $L^2(\mathbb{R}^n)$; in particular, (2.15) and (2.16) must hold with $\varphi^i = \psi^i$ for $1 \leqslant i \leqslant L$.

The generalization to wavelet systems and the reversal of the order of dilation and translation in passing from affine systems to quasi-affine systems raise further questions, and there is reason to hope this may be elucidated by the less technically formidable investigation of continuous wavelets for subgroups of $GL(n, \mathbb{R})$. This is an area of active research which we shall not comment upon further in these notes. Instead, we turn to the techniques needed to prove the new characterization of dyadic wavelets announced in Proposition III in the first section and the generalized Proposition III$'$.

3. Shift Invariant Subspaces and a New Characterization of Wavelets

Proposition III was stated as a conjecture by the first author in a seminar. Two students, M. Bownik and Z. Rzeszotnik, proved it independently. We present the n-dimensional extension of an argument in the Ph.D. thesis of the latter [R]; for a different approach see [B_2].

Suppose φ is a non-zero function in $L^2(\mathbb{R}^n)$. Let $\mathcal{A}_\varphi$ denote the algebraic span of the translates $\varphi(\cdot - k) = \tau_k\varphi$ where $k \in \mathbb{Z}^n$. That is, $\mathcal{A}_\varphi$ is the linear space of all finite linear combinations of the translates $\tau_k\varphi$. Let

$$\omega_\varphi(\xi) = \sum_{l\in\mathbb{Z}^n} |\widehat{\varphi}(\xi + l)|^2. \tag{3.1}$$

It is clear that ω_φ is a 1-periodic function that is integrable on $\mathbb{T}^n = \{\xi = (\xi_1, \xi_2, ..., \xi_n) : 0 \leqslant \xi_j < 1,\ j = 1, 2, ..., n\}$ (by "1-periodic" we mean that it is 1-periodic in each variable). Moreover, if $f = \sum_{finite} a_k\tau_k\varphi$ is the general element of $\mathcal{A}_\varphi$, then

$$\widehat{f}(\xi) = \left\{\sum_{finite} a_k e^{-2\pi ik\cdot\xi}\right\}\widehat{\varphi}(\xi) = t(\xi)\widehat{\varphi}(\xi).$$

Conversely, if $\widehat{f}(\xi) = t(\xi)\,\widehat{\varphi}(\xi)$, with t a trigonometric polynomial, then $f \in \mathcal{A}_\varphi$. Thus, for such an f we have

$$||f||_2^2 = ||\widehat{f}||_2^2 = \int_{\mathbb{R}^n} |t(\xi)|^2\,|\widehat{\varphi}(\xi)|^2\,d\xi = \sum_{k\in\mathbb{Z}^n}\int_{\tau_k\mathbb{T}^n} |t(\xi)|^2\,|\widehat{\varphi}(\xi)|^2\,d\xi =$$

$$\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n} |t(\xi+k)|^2\,|\widehat{\varphi}(\xi+k)|^2\,d\xi = \int_{\mathbb{T}^n} |t(\xi)|^2\,\omega_\varphi(\xi)d\xi.$$

This shows that the mapping $\widetilde{U}_\varphi$ that assigns to $f \in \mathcal{A}_\varphi$ the unique trigonometric polynomial t such that $\widehat{f} = t\widehat{\varphi}$ is an isometry between $\mathcal{A}_\varphi$ and the space P_φ of all trigonometric polynomials endowed with the norm

$$||t||_{L^2(\mathbb{T}^n,\omega_\varphi)} = \left(\int_{\mathbb{T}^n} |t(\xi)|^2\,\omega_\varphi(\xi)\,d\xi\right)^{1/2}.$$

Thus, $\widetilde{U}_\varphi$ has a unique extension to an isometry U_φ between V_φ, the closure of $\mathcal{A}_\varphi$ in $L^2(\mathbb{R}^n)$, and the space $L^2(\mathbb{T}^n, \omega_\varphi)$ consisting of all 1-periodic functions s satisfying $||s||_{L^2(\mathbb{T}^n,\omega_\varphi)} < \infty$. Observe that, as functions on $\mathbb{T}^n$ (or $\mathbb{R}^n$ and 1-periodic), two elements that are equal on $\Omega_\varphi = \{\xi : \omega_\varphi(\xi) \neq 0\}$ represent the same element of $L^2(\mathbb{T}^n, \omega_\varphi)$.

When $f \in V_\varphi$, then $U_\varphi f = s$ with $s \in L^2(\mathbb{T}^n, \omega_\varphi)$ and

$$(3.2) \qquad ||f||_2 = ||s||_{L^2(\mathbb{T},\omega_\varphi)},$$

where $\widehat{f} = s\,\widehat{\varphi}$. Conversely, any $s\,\epsilon L^2(\mathbb{T}^n, \omega_\varphi)$ gives rise to an $f\,\epsilon\,V_\varphi$ via the last equality.

The translates of $\varphi, \tau_k\,\varphi, k\,\epsilon\,\mathbb{Z}^n$ generate V_φ in the manner just described. In view of the notions we have been discussing, it is natural to ask if V_φ, which is clearly shift invariant, contains a θ which also generates V_φ and $\{\tau_k\,\theta\}, k\,\epsilon\,\mathbb{Z}^n$, is a tight frame for this subspace? The answer is "yes" and is easily obtained: Let $s(\xi) = \omega_\varphi\,(\xi)^{-1/2}$ for $\xi\,\epsilon\,\Omega_\varphi$ and, say, $s(\xi) = 0$ outside Ω_φ. It is easily seen that $\widehat{\theta}\,(\xi) = s(\xi)\widehat{\varphi}(\xi)$ gives a function $\theta\,\epsilon\,L^2(\mathbb{R}^n)$ having these properties: obviously $s\,\epsilon\,L^2(\mathbb{T}^n, \omega_\varphi)$; in fact,

$$||s||^2_{L^2(\mathbb{T}^n,\omega_\varphi)} = \int_{\mathbb{T}^n} X_{\Omega_\varphi}\,(\xi)d\,\xi = |\Omega_\varphi| \leqslant 1.$$

If $f\,\epsilon\,V_\varphi$ so that $\widehat{f} = t\widehat{\varphi}$ and $\widehat{\theta} = s\,\widehat{\varphi}$, then $\{\theta_k\} = \{\tau_k\,\theta\}$ is a tight frame for V_φ if and only if

$$(3.3) \qquad \sum_{k\,\epsilon\mathbb{Z}^n} |\langle f, \theta_k\rangle|^2 = ||f||_2^2.$$

But, using Plancherel's theorem, and "periodizing" the integral over $\mathbb{R}^n$,

$$\langle f, \theta_k\rangle = \int_{\mathbb{R}_n} t(\xi)\widehat{\varphi}(\xi)e^{2\,\pi\,ik\cdot\xi}\overline{s(\xi)\widehat{\varphi}(\xi)}d\,\xi =$$

$$\int_{\mathbb{T}^n} t(\xi)\overline{s(\xi)}e^{2\,\pi\,ik\cdot\xi} \sum_{l\,\epsilon\mathbb{Z}^n} |\widehat{\varphi}(\xi + l)|^2\,d\,\xi = \int_{\mathbb{T}^n} t(\xi)\overline{s(\xi)}\omega_\varphi(\xi)e^{2\,\pi\,ik\cdot\xi}\,d\,\xi.$$

Thus, $\{\langle f, \theta_k\rangle\}, k\,\epsilon\mathbb{Z}^n$, is the sequence of Fourier coefficients of the function $t(\xi)\overline{s(\xi)}\omega_\varphi(\xi)$. Thus, by this calculation and (3.2),

$$(3.4) \qquad \sum_{k\,\epsilon\,\mathbb{Z}^n} |\langle f, \theta_k\rangle|^2 = \int_{\mathbb{T}^n} |t(\xi)|^2\,|s(\xi)|^2\,(\omega_\varphi(\xi))^2\,d\,\xi = ||f||_2^2.$$

This proves (3.3)

On the other hand, by (3.2) (with s replaced by t),

$$||f||_2^2 = \int_{\mathbb{T}^n} |t(\xi)|^2\,\omega_\varphi(\xi)d\xi.$$

From this equality, (3.3) and (3.4) we have

$$(3.5) \qquad 0 = \int_{\mathbb{T}^n} |t(\xi)|^2\,\omega_\varphi(\xi)[1 - |s(\xi)|^2\,\omega_\varphi\,(\xi)]\,d\,\xi$$

for all $t\,\epsilon\,L^2(\mathbb{T}^n,\omega_\varphi)$. Choosing $t=\chi_E$ where E is either $\{\xi\epsilon\,\Omega_\varphi : 1 > |s(\xi)|^2\,\omega_\varphi\,(\xi)\}$ or $\{\xi\,\epsilon\,\Omega_\varphi : 1 < |s(\xi)|^2\,\omega_\varphi\,(\xi)\}$ we see that (3.5) is equivalent to

$$1-|s(\xi)|^2\,\omega_\varphi\,(\xi)=0$$

for a.e. $\xi\,\epsilon\,\Omega_\varphi$. We have proved

LEMMA (3.6). *For each space $V_\varphi, \varphi\,\epsilon\,L^2(\mathbb{R}^n)$, we can find $\theta\,\epsilon\,V_\varphi$ such that*

$$\{\theta(\cdot-k)\}, k\,\epsilon\,\mathbb{Z}^n,$$

is a tight frame (of constant 1) for V_φ. All such θ are characterized by having Fourier transforms $\widehat{\theta}(\xi)=\nu(\xi)\,\omega_\varphi\,(\xi)^{-1/2}\,\widehat{\varphi}\,(\xi)$, where ν is a 1-periodic unimodular function. Moreover, the tight frame property of these θ is characterized by the equality

(3.7) $$\sum_{k\,\epsilon\,\mathbb{Z}^n}|\widehat{\theta}(\xi+k)|^2$$

(3.8) $$=\mathcal{X}_{\Omega_\varphi}\,(\xi)\ \text{ a.e.}\mathit{in}\,\mathbb{R}^n.$$

The elements of $V_\varphi=V_\theta$ are precisely those whose Fourier transform is of the form

$$t(\xi)\,\widehat{\theta}\,(\xi), t\epsilon\,L^2(\mathbb{T}^n,d\,\xi).$$

Remark. Equality (3.7) is, clearly, a more general version of equality (A) in Proposition I that characterizes the orthonormality of the system

$$\{\psi(\cdot-k)\}, k\,\epsilon\,\mathbb{Z}^n.$$

From this lemma we obtain the following characterization of shift invariant subspaces:

THEOREM (3.1). *Suppose V is a closed subspace of $L^2(\mathbb{R}^n)$. V is shift invariant if and only if there exists a sequence of functions $\{\theta^j\}, 1\leqslant j$, belonging to V that are mutually orthogonal such that each θ^j generates a tight frame (of constant 1), $\{\theta^j(\cdot-k)\}, k\,\epsilon\,\mathbb{Z}^n$ for the space V_{θ^j} and*

(3.9) $$V=\oplus_{j=1}^{\infty}V_{\theta^j}$$ [1]

Remark. All but a finite number of the θ^j can be the zero function; in this case $V_{\theta^j}=\{0\}$. Unless V is the trivial space $\{0\}$, let us order the θ^j so that the non-zero ones are listed at the beginning.

[1] The symbol $\oplus_{j=1}^{\infty}$ denotes the orthogonal direct sum of the sequence of subspaces that follows it.

Proof. It is clear that if V satisfies (3.9) then it is shift invariant. Thus, we only need to show that a shift invariant closed subspace V satisifes (3.9). Toward this end we choose a non-zero $\varphi \,\epsilon\, V$ (if such φ exists) and apply Lemma (3.6) to obtain a $\theta \,\epsilon\, V_\varphi$ satisfying (3.7). We let $\theta^1 = \theta$ and consider the orthogonal complement of V_{θ^1} in V. Applying the same argument to $V \cap V_{\theta^1}^{\perp}$ we obtain θ^2 in this orthogonal complement. Continuing in this fashion we obtain (3.9) (if we wish to be completely rigorous, we invoke the separability of V and Zorn's lemma).

Now suppose V is shift invariant and, thus, equals a direct sum as in (3.9). Fix $j \geqslant 1$ and $\xi \,\epsilon\, \mathbb{R}^n$. Let $\Theta^j(\xi)$ be the vector in $l^2(\mathbb{Z}^n)$ whose k^{th} coordinate is $\widehat{\theta^j}(\xi + k), k \,\epsilon\, \mathbb{Z}^n$. Since $\{\theta^j(\cdot - k)\}_{k \,\epsilon\, \mathbb{Z}^n}$ is a tight frame, equality (3.7) tells us that

$$(3.10) \qquad ||\Theta^j(\xi)||^2_{l^2(\mathbb{Z}^n)} = \sum_{k \,\epsilon\, \mathbb{Z}^n} |\widehat{\theta^j}(\xi + k)|^2 = \mathcal{X}_{\Omega_{\theta^j}}(\xi) = 0 \text{ or } 1$$

(in order to avoid having to repeatedly add the expression "a.e." we tacitly assume that we only choose ξ in a subset of $\mathbb{R}^n$ whose complement has measure 0 and, for all such ξ, (3.7) and the other related properties we invoke are valid).

The orthogonality of the spaces V_{θ^j}, and a periodization argument like the one that gives us equalities (A) and (B), yields

$$(3.11) \qquad \langle\, \Theta^j(\xi), \Theta^{j'}(\xi)\,\rangle_{l^2(\mathbb{Z}^n)} = 0$$

if $j \neq j'$.

Let $\mathcal{L}(\xi)$ be the closure in $l^2(\mathbb{Z}^n)$ of the linear space generated by the vectors $\Theta^j(\xi), 1 \leqslant j$. It is an immediate consequence of (3.10) and (3.11) that the sequence $\{\Theta^j(\xi)\}, 1 \leqslant j$ is a tight frame (of constant 1) for $\mathcal{L}(\xi)$ (even if $\mathcal{L}(\xi) = \{0\}$). Let $P(\xi)$ be the orthogonal projection of $l^2(\mathbb{Z}^n)$ onto $\mathcal{L}(\xi)$ and $e_0 \,\epsilon\, l^2(\mathbb{Z}^n)$ the vector all of whose coordinates are 0 except for the coordinate corresponding to $0 \,\epsilon\, \mathbb{Z}^n$, which has value $1 : e_0(k) = 0$ if $k \neq 0$ and $e_0(0) = 1$. Then, using the tight frame property for $\{\Theta^j(\xi)\}$ we have

$$1 \geqslant ||P(\xi)e_0||^2_{l^2(\mathbb{Z}^n)} = \sum_{j \geqslant 1} |\langle P(\xi)e_0, \, \Theta^j(\xi)\rangle|^2 =$$

$$\sum_{j \geqslant 1} |\langle e_0, P(\xi)\,\Theta^j(\xi)\rangle|^2 = \sum_{j \geqslant 1} |\langle e_0, \Theta^j(\xi)\rangle|^2 =$$

$$\sum_{j \geqslant 1} |\widehat{\theta^j}(\xi)|^2 \equiv \sigma(\xi)\,(= \sigma_V(\xi)).$$

This shows

LEMMA (3.11). *If V is a closed shift invariant subspace of $L^2(\mathbb{R}^n)$ and we represent V as the orthogonal direct sum (3.9), then*

(3.12)
$$\sigma_V(\xi) = \sum_{j \geqslant 1} |\widehat{\theta}^j(\xi)|^2 \leqslant 1.$$

The following result will be an essential tool we shall use in the proof of the principal result of this section. An interesting feature is that the dyadic dilation operator D arises naturally in this study of the properties associated with the translation operators τ_k.

LEMMA (3.13). *Suppose we have the same hypothesis as in the previous lemma. If $\sigma_V \,\epsilon\, L^1(\mathbb{R}^n)$, then*

(3.14)
$$\cap_{j \,\epsilon\, \mathbb{Z}} D^j V = \{0\}.$$

Proof. Suppose there exists a non-zero $f \,\epsilon\, \cap_{j \,\epsilon\, \mathbb{Z}} D^j V$; we might as well assume $||f||_2 = 1$. Since

$$f \,\epsilon\, \cap_{j \geqslant 0} D^{-j} V$$

we must have $D^j f \,\epsilon\, V$ for $j \geqslant 0$.

If $g \,\epsilon\, V$ then, by Lemma (3.6) and equality (3.8),

$$\hat{g}(\xi) = \sum_{j=1}^{\infty} m^j(\xi)\, \widehat{\theta}^j(\xi)$$

(convergence is in $L^2(\mathbb{R}^n)$, where each m^j is a 1-periodic function in $L^2(\mathbb{T}^n \cap \Omega_{\theta^j})$ that is uniquely determined on Ω_{θ^j}). Moreover,

(3.15)
$$||\hat{g}||_2^2 = \sum_{j=1}^{\infty} ||m^j||^2_{L^2(\mathbb{T}^n \cap \Omega_{\theta^j})},$$

(by (3.2) since, in this case, $\omega_\varphi = \mathcal{X}_{\Omega_{\theta^j}}$). In particular, applying the above equality for $\hat{g}(\xi)$ and (3.15) to $g = D^l f, l \geqslant 0$,

$$2^{-nl/2}\, \hat{f}(2^{-l}\,\xi) = \sum_{j=1}^{\infty} m_l^j(\xi) \widehat{\theta}^j(\xi),$$

where m_l^j is 1-periodic, supported in Ω_{θ^j} and

(3.16)
$$1 = ||D^l f||_2^2 = \sum_{j=1}^{\infty} ||m_l^j||^2_{L^2(\mathbb{T}^n \cap \Omega_{\theta^j})} = \sum_{j=1}^{\infty} ||m_l^j||^2_{L^2(\mathbb{T}^n)}.$$

Therefore, if $l \geqslant 0$ we have

$$\hat{f}(\xi) = 2^{nl/2} \sum_{j=1}^{\infty} m_l^j(2^l \xi)\, \widehat{\theta^j}(2^l \xi). \tag{3.17}$$

Let $\mathcal{Q}$ be the translate of $\mathbb{T}^n = \{\xi = (\xi_1, \xi_2, \ldots, \xi_n) : 0 \leqslant \xi_j < 1, j = 1, 2, \ldots, n\}$ by $k = (k_1, k_2, \ldots, k_n) \,\epsilon\, \mathbb{Z}^n$ satisfying $k_1, k_2, \ldots, k_n \geqslant 1$. We claim that $\hat{f}(\xi) = 0$ for $\xi \,\epsilon\, \mathcal{Q}$. To see this we first observe that

$$\begin{aligned} &\textstyle\int_{\mathcal{Q}} \sum_{j=1}^{\infty} |m_l^j(2^l \xi)|^2 \, d\xi = 2^{-nl} \int_{2^l \mathcal{Q}} \sum_{j=1}^{\infty} |m_l^j(\xi)|^2 \, d\xi = \\ &\textstyle 2^{-nl}\, 2^{nl} \int_{\mathbb{T}^n} \sum_{j=1}^{\infty} |m_l^j(\xi)|^2 \, d\xi = 1. \end{aligned} \tag{3.18}$$

This is a consequence of the fact that $2^l\mathcal{Q}$ is the disjoint union of 2^{nl} lattice point translates of $\mathbb{T}^n$, the 1-periodicity of m_l^j, and (3.16). Hence, using (3.17) and (3.18),

$$\begin{gathered} \int_{\mathcal{Q}} |\hat{f}(\xi)| \, d\xi \;\leqslant 2^{nl/2} \sum_{j=1}^{\infty} \int_{\mathcal{Q}} |m_l^j(2^l \xi)| \, |\widehat{\theta^j}(2^l \xi)| d\xi \;\leqslant \\ 2^{nl/2} \int_{\mathcal{Q}} \left(\sum_{j=1}^{\infty} |\widehat{\theta^j}(2^l \xi)|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |m_l^j(2^l \xi)|^2 d\xi \right)^{1/2} d\xi \;\leqslant \\ 2^{nl/2} \left(2^{-nl} \int_{2^l \mathcal{Q}} \sum_{j=1}^{\infty} |\widehat{\theta^j}\,(\eta)|^2 d\eta \right)^{1/2} \left(2^{-nl} \int_{2^l \mathcal{Q}} \sum_{j=1}^{\infty} m_l^j(\eta)|^2 d\eta \right)^{1/2} \leqslant \\ \left(\int_{2^l \mathcal{Q}} \sigma_V(\eta) d\eta \right)^{1/2} \cdot 1. \end{gathered}$$

But the last expression tends to 0 as $l \longrightarrow \infty$ since σ_V is integrable and the points η of $2^l\mathcal{Q}$ satisfy $2^l\sqrt{n} \leqslant 2^l|k| \leqslant |\eta|$. This shows that $f(\xi) = 0$ when $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ with $\xi_j \geqslant 1$ for $j = 1, 2, \ldots, n$.

Since $D^l f \,\epsilon\, V$ for $l \geqslant 1$ we can apply this argument to $\hat{f}(2^{-l}\,\xi)$ to obtain the fact that $\hat{f}(\xi)$ vanishes in the entire first quadrant. It is also easy to see that this proof, with obvious changes, shows that $\hat{f}$ vanishes in the remaining quadrants. Hence, $\hat{f}$ and f vanish a.e., contradicting the assumption $f \neq 0$.

We are now ready to establish the main result of this section:

THEOREM (3.2). *Suppose $\psi \,\epsilon\, L^2(\mathbb{R}^n)$ and the system $\{\psi_{jk}\}, j \,\epsilon\, \mathbb{Z}, k \,\epsilon\, \mathbb{Z}^n$ is orthonormal, then this system is an orthonormal basis for $L^2(\mathbb{R}^n)$ if and only if ψ satisfies equality (C).*

This means that equalities (A), (B) and (C) in section 1 completely characterize all wavelets.

Proof. For each $j \in \mathbb{Z}$ let

$$W_j = \overline{span\{\psi_{jk} : k \in \mathbb{Z}^n\}},$$

where we assume that the system $\{\psi_{jk}\}, j \in \mathbb{Z}, k \in \mathbb{Z}^n$ is orthonormal. As we explained above (see Proposition III) if ψ is a wavelet, then ψ satisfies (C). Thus, all we need to do is to show that, under our assumptions, if (C) is satisfied then ψ is a wavelet.

Clearly, $W_j \perp W_{j'}$ when $j \neq j'$. Thus,

$$L^2(\mathbb{R}^n) = \{\oplus_{j \in \mathbb{Z}} W_j\} \oplus \{\oplus_{j \in \mathbb{Z}} W_j\}^\perp$$

and we must show that $\{\oplus_{j \in \mathbb{Z}} W_j\}^\perp = \{0\}$. Let

$$V = \{\oplus_{j \geqslant 0} W_j\}^\perp.$$

Since each $W_j, j \geqslant 0$, is shift invariant, it follows that V is shift invariant. We claim that

$$\{\oplus_{j \in \mathbb{Z}} W_j\}^\perp \subset \cap_{l \in \mathbb{Z}} D^l V. \tag{3.19}$$

This follows from the fact that $D^l V = \{\oplus_{j \geqslant l} W_j\}^\perp \supset \{\oplus_{j \in \mathbb{Z}} W_j\}^\perp$. Thus, it is natural to see if Lemma (3.13) can be applied in order to conclude that $\cap_{l \in \mathbb{Z}} D^l V = \{0\}$. We have, by (3.9),

$$V = \oplus_{j=1}^{\infty} V_{\theta^j}, \tag{3.20}$$

where the functions θ^j satisfy the properties described in Theorem (3.1). We want to show that $\sigma_V(\xi) = \sum_{j=1}^{\infty} |\widehat{\theta^j}(\xi)|^2$ is integrable. In fact, we shall apply (3.9) to

$$L^2(\mathbb{R}^n) = V \oplus \{\oplus_{l \geqslant 0} W_l\} \tag{3.21}$$

which is clearly shift invariant. Given $q \in \mathbb{Z}^n$ and $l \geqslant 0$ there exist unique $k, p \in \mathbb{Z}^n$ with $p = (p_1, p_2, \ldots, p_n)$ satisfying $0 \leqslant p_j \leqslant 2^l - 1, j = 1, 2, \ldots, n$, such that $q = 2^l k + p$. There exist precisely 2^{nl} such p and each of these lattice points determines the orthonormal system

$$\{\psi_{lq}(x)\} = \{2^{nl/2}\, \psi(2^l(x-k) - p)\}, k \in \mathbb{Z}^n,$$

which is generated by the lattice point translations of the function $\varphi_p^l(x) = 2^{nl/2}\, \psi(2^l x - p)$. That is, each space $W_l, l \geqslant 0$, which is shift invariant, has the special (3.8) decomposition

$$W_l = \oplus_p V_{\varphi_p^l}. \tag{3.22}$$

Putting together (3.20), (3.21) and (3.22), therefore, we have

$$L^2(\mathbb{R}^n) = \{\oplus_{j=1}^{\infty} V_{\theta^j}\} \oplus \{\oplus_{l \geqslant 0}\{\oplus_p V_{\varphi_p^l}\}\}.$$

Applying Lemma (3.11) we then must have

$$\sum_{j=1}^{\infty} |\widehat{\theta}^j(\xi)|^2 + \sum_{l=0}^{\infty} \sum_{p} |\widehat{\varphi}_p^l(\xi)|^2 =$$
$$\sigma_V(\xi) + \sum_{l=0}^{\infty} \sum_{p} |\widehat{\psi}_{lp}(\xi)|^2 \leqslant 1.$$

But $\widehat{\psi}_{lp}(\xi) = 2^{-nl/2} e^{-2\pi i 2^{-l} \xi \cdot p} \widehat{\psi}(2^{-l}\xi)$ and, thus,

$$|\widehat{\psi_{lp}}(\xi)| = 2^{-nl/2} |\widehat{\psi}(2^{-l}\xi)|$$

Consequently, the sum $\sum_p |\widehat{\psi}_{lp}(\xi)|^2$ consists of 2^{nl} equal terms and, thus, must be equal to $|\widehat{\psi}(2^{-l}\xi)|^2$. We conclude that

$$1 \geqslant \sigma_V(\xi) + \sum_{l=0}^{\infty} \sum_{p} |\widehat{\psi}_{lp}(\xi)|^2 = \sigma_V(\xi) + \sum_{l=0}^{\infty} |\widehat{\psi}(2^{-l}\xi)|^2.$$

Because of (C), therefore, $\sigma_V(\xi) \leqslant 1 - \sum_{l=0}^{\infty} |\widehat{\psi}(2^{-l}\xi)|^2 = \sum_{l=1}^{\infty} |\widehat{\psi}(2^{l}\xi)|^2$. Hence,

$$\int_{\mathbb{R}^n} \sigma_V(\xi) d\xi \leqslant \int_{\mathbb{R}^n} \sum_{l=1}^{\infty} |\widehat{\psi}(2^l\xi)|^2 \, d\xi = \left(\sum_{l=1}^{\infty} 2^{-nl} \right) ||\psi||_2^2 = \frac{1}{2^n - 1} < \infty.$$

This shows that σ_V is integrable and Theorem (3.2) is established.

Before ending this section let us make some observations. We introduced the characterization of systems $\Psi = \{\psi^1, \dots, \psi^L\}$ that generate tight frames in the discussion following Proposition IV. Theorem (3.2) was stated and proved for the case L=1. As we stated before, we did this for simplicity. If we assume the orthonormality of the system $\{\psi_{jk}^l\}, l = 1, \dots, L, 1 \leqslant j, k \,\epsilon\, \mathbb{Z}^n$, it is easy to see that our proof goes through when equality (C) is satisfied. It is a curious fact that if we assume that the functions ψ^l have norm at least 1 then (C) and (D) are equivalent to the property that the system $\{\psi_{jk}^l\}$ is an orthonormal basis for L^2. We need not assume that the L functions $\psi^l, l = 1, \dots, L,$ are mutually orthogonal. This orthogonality is a consequence of (C) and (D).

One can extend Theorem (3.2) to tight frames that are semiorthogonal (that is the subspaces W_j are mutually orthogonal and the system $\{\psi_{jk}^l\}$ is a tight frame for the subspace it generates).

The characterization of shift invariant subspaces, at least in the case of finitely many subspaces V_θ, was introduced in [RS]. The general case was also obtained by M. Bownik [B_2] using a different approach.

4. Multiresolution Analyses in $\mathbb{R}^n$

A *multiresolution analysis* (MRA) is a sequence of subspaces $\{V_j\}, j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ satisfying the following conditions

(4.1) $$V_j \subset V_{j+1}, j \in \mathbb{Z},$$

(4.2) $$f \in V_j \text{ if and only if } f(2\cdot) \in V_{j+1}, j \in \mathbb{Z},$$

(4.3) $$\overline{\cup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n),$$

(4.4) there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}, k \in \mathbb{Z}^n$ is an orthonormal basis for V_0.

The function φ in (4.4) is called a *scaling function* for this MRA.

It is not hard to show that (4.1), (4.2) and (4.4) imply

(4.5) $$\cap_{j \in \mathbb{Z}} V_j = \{0\}.$$

The proof of the one-dimensional version of this implication can be found on page 45 of [HW]. The argument given there is very similar to the one we presented in Lemma (3.13). In fact, we adapted the argument in [HW] to provide the proof of (3.13).

The construction of a wavelet basis from an MRA can be described in the following way: For each $j \in \mathbb{Z}$ let $W_j = V_{j+1} \cap V_j^{\perp}$. A consequence of the above hypotheses is that these spaces are mutually orthogonal with

(4.6) $$V_j \oplus W_j = V_{j+1}, j \in \mathbb{Z},$$

and

(4.7) $$L^2(\mathbb{R}^n) = \oplus_{j \in \mathbb{Z}} W_j.$$

If we can find a system $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset W_0$ such that

$$\{\psi^l(\cdot - k)\}, l = 1, 2, \ldots, L, k \in \mathbb{Z}^n$$

is an orthonormal basis for W_0, then (4.2), (4.6) and (4.7) imply that

$$\{\psi^l_{jk}\}, l = 1, 2, \ldots, L, j \in \mathbb{Z}, k \in \mathbb{Z}^n,$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$. That is, Ψ generates a wavelet basis for $L^2(\mathbb{R}^n)$.

It is convenient to express these properties in terms of the Fourier transform. By doing so we shall see that the construction of appropriate systems Ψ raises some interesting problems and, in particular, we will discover that L must equal $2^n - 1$. It is clear that

$$\widehat{V}_0 = \{\hat{f} : f \,\epsilon\, V_0\} = \{\hat{f}(\xi) = m(\xi)\widehat{\varphi}(\xi) : m \,\epsilon\, L^2(\mathbb{T}^n)\}$$

(where m is 1-periodic and its Fourier coefficients are the coefficients in the expansion $f = \sum_{k\,\epsilon\,\mathbb{Z}^n} \alpha_k\, \varphi(\cdot - k)$, that is provided by property (4.4)).

The elements of $\widehat{V}_0$, by (4.2), also provide us with those of $\widehat{V}_{-1}$ and $\widehat{W}_{-1}$. In each case they are of the form $\widehat{\theta}(2\,\xi)$ with $\theta \,\epsilon\, V_0$; moreover, we must have

$$\widehat{\theta}(2\,\xi) = m(\xi)\widehat{\varphi}(\xi) \tag{4.8}$$

and, since φ satisfies (A),

$$2^{-n}||\widehat{\theta}||^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |m(\xi)|^2\, |\widehat{\varphi}(\xi)|^2 d\,\xi = \int_{\mathbb{T}^n} |m(\xi)|^2 \sum_{k\,\epsilon\mathbb{Z}^n} |\widehat{\varphi}(\xi + k)|^2\, d\,\xi$$
$$= \int_{\mathbb{T}^n} |m(\xi)|^2\, d\,\xi.$$

By polarization, therefore,

$$2^{-n}\langle\widehat{\theta}, \widehat{\varphi}\rangle_{L^2(\mathbb{R}^n)} = \langle t, s\rangle_{L^2(\mathbb{T}^n)} \tag{4.9}$$

whenever $\widehat{\theta}(2\,\xi) = t(\xi)\widehat{\varphi}(\xi)$ and $\widehat{\psi}(2\,\xi) = s(\xi)\widehat{\varphi}(\xi)$ are representations of $\theta, \psi\, \epsilon V_0$ via equality (4.8). We shall examine some consequences of (4.8) and (4.9) when $\theta = \varphi$ and $\psi\,\epsilon\, W_0$ is such that the system $\{\psi(\cdot - k)\}, k\,\epsilon\,\mathbb{Z}^n$, is orthonormal. It will be useful to consider the set of vertices of $\mathbb{T}^n$:

$$\vartheta_n = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) : \varepsilon_j = 0 \text{ or } 1, 1 \leqslant j \leqslant n\}.$$

By (4.8) we have $\widehat{\varphi}(2\,\xi) = m_0(\xi)\widehat{\varphi}(\xi)$ and since φ satisfies equality (A) (because the translates by $k\,\epsilon\,\mathbb{Z}^n$ of φ form an orthonormal system) we have

$$1 = \sum_{k\,\epsilon\,\mathbb{Z}^n} |\widehat{\varphi}(2\,\xi + k)|^2 = \sum_{k\,\epsilon\,\mathbb{Z}} |m_0(\xi + \frac{1}{2}\,k)|^2\, |\widehat{\varphi}(\xi + \frac{1}{2}\,k)|^2$$
$$= \sum_{\varepsilon\,\epsilon\,\vartheta_n} |m_0(\xi + \frac{1}{2}\varepsilon)|^2 \sum_{l\,\epsilon\,\mathbb{Z}^n} |\widehat{\varphi}(\xi + \frac{\varepsilon}{2} + l)|^2 = \sum_{\varepsilon\,\epsilon\,\vartheta_n} |m_0(\xi + \frac{1}{2}\,\varepsilon)|^2.$$

We have used the 1-periodicity on m_0 and the fact that to each $k\,\epsilon\,\mathbb{Z}^n$ there exists a unique $l\,\epsilon\,\mathbb{Z}^n$ and $\varepsilon\,\epsilon\,\vartheta_n$ such that $(1/2)k = l + (1/2)\varepsilon$. This shows that the 2^n-dimensional vector with components $m_0(\xi + (1/2)\varepsilon)$ has norm 1 (for a.e. $\xi\,\epsilon\,\mathbb{R}^n$):

$$1 = \sum_{\varepsilon \,\epsilon\, \vartheta_n} |m_0(\xi + \frac{1}{2}\,\varepsilon)|^2. \tag{4.10}$$

Since we are assuming $\psi \,\epsilon\, W_0$ also satisfies (A) this shows that the "filter" m defined by the equality $\widehat{\psi}(2\,\xi) = m(\xi)\widehat{\varphi}(\xi)$ also satisfies (4.10). [2]

We are also assuming that φ and $\psi(\cdot - k)$ are orthogonal for all $k \,\epsilon\, \mathbb{Z}^n$. Hence,

$$0 = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}e^{2\,\pi\, ik\cdot\xi}\, d\,\xi = \int_{\mathbb{T}^n} \left\{ \sum_{l\,\epsilon\,\mathbb{Z}^n} \widehat{\varphi}(\xi + l)\overline{\widehat{\psi}(\xi + l)} \right\} e^{2\,\pi\, ik\cdot\xi}\, d\,\xi.$$

That is, all Fourier coefficients of the 1-periodic function $\sum_{l\,\epsilon\,\mathbb{Z}^n} \widehat{\varphi}(\xi + l)\overline{\widehat{\psi}(\xi + l)}$ are 0. Thus for a.e. $\xi \,\epsilon\, \mathbb{R}^n$

$$0 = \sum_{k\,\epsilon\,\mathbb{Z}^n} \widehat{\varphi}(\xi + k)\overline{\widehat{\psi}(\xi + k)}. \tag{4.11}$$

Consequently,

$$\begin{gathered} 0 = \sum_{k\,\epsilon\,\mathbb{Z}^n} \widehat{\varphi}(2\,\xi + k)\overline{\widehat{\psi}(2\,\xi + k)} = \\ \sum_{k\,\epsilon\,\mathbb{Z}^n} m_0(\xi + \frac{1}{2}k)\overline{m(\xi + \frac{1}{2}\,k)}\,|\widehat{\varphi}(\xi + \frac{1}{2}\,k)|^2 = \\ \sum_{\varepsilon\,\epsilon\,\vartheta_n} m_0(\xi + \frac{1}{2}\,\varepsilon)\overline{m(\xi + \frac{1}{2}\,\varepsilon)} \sum_{l\,\epsilon\,\mathbb{Z}^n} |\widehat{\varphi}(\xi + \frac{1}{2}\,\varepsilon + l)|^2 = \\ \sum_{\varepsilon\,\epsilon\,\vartheta_n} m_0(\xi + \frac{1}{2}\,\varepsilon)\overline{m(\xi + \frac{1}{2}\,\varepsilon)}. \end{gathered}$$

This shows that the 2^n-dimensional vectors $\{m_0(\xi + \frac{1}{2}\,\varepsilon)\}$ and $\{m(\xi + \frac{1}{2}\,\varepsilon)\}, \varepsilon \,\epsilon\, \vartheta_n$, are orthogonal to each other:

$$0 = \sum_{\varepsilon\,\epsilon\,\vartheta_n} m_0(\xi + \frac{1}{2}\,\varepsilon)\overline{m(\xi + \frac{1}{2}\,\varepsilon)}. \tag{4.12}$$

In fact, these properties characterize the "wavelets" $\psi \,\epsilon\, W_0$: $\psi \,\epsilon\, W_0$ is such that the system $\{\psi(\cdot - k)\}, k \,\epsilon\, \mathbb{Z}_n$, is orthonormal if and only if the vector $\{m(\xi + \frac{1}{2}\,\varepsilon)\}, \varepsilon \,\epsilon\, \vartheta_n$, has norm 1,

[2] In general, $f \,\epsilon\, V_0$ iff $\hat{f}(\xi) = t(\xi)\widehat{\varphi}(\xi)$ with t 1-periodic and in $L^2(\mathbb{T}^n)$. We shall call t the *filter* associated with f. It is unique.

as in (4.10), and satisfies (4.12) (just reverse the order of the sequence in the equalities that established (4.10) and (4.12)).

Let us use the notation $\psi^0 = \varphi$ and $\psi^1 = \psi$ (for a $\psi \,\epsilon\, W_0$ having the properties we just discussed). Suppose we find $\psi^l, l = 2, 3, \ldots, L$, in W_0 with filters m_l (that is, $\widehat{\psi}^l(2\,\xi) = m_l(\xi)\widehat{\varphi}(\xi)\epsilon\, L^2(\mathbb{T}^n), m_l$ 1-periodic) such that

$$\text{(4.13)} \qquad \sum_{\varepsilon\,\epsilon\,\vartheta_n} m_{l_1}(\xi + \frac{1}{2}\,\varepsilon)\overline{m_{l_2}(\xi + \frac{1}{2}\,\varepsilon)} = \delta_{l_1 l_2}$$

a.e., then the collection $\Psi = \{\psi^1, \psi^2, \ldots \psi^L\}$ generates the orthonormal system $\{\psi^l_{j,k}\}, l = 1, 2, \ldots, L, j\,\epsilon\,\mathbb{Z}, k\,\epsilon\,\mathbb{Z}^n$. It is clear that L cannot exceed $2^n - 1$ since the $(L+1) \times 2^n$ matrix has row vectors $\{m_l(\xi + \frac{1}{2}\,\varepsilon)\}, \varepsilon\,\epsilon\,\vartheta_n, 0 \leqslant l \leqslant L$, that satisfy (4.13); that is, the $L + 1$ row vectors form an orthonormal system for a.e. ξ (as the last calculation in this section shows, we must have $L = 2^n - 1$). Consequently, we are presented with the following question: given the MRA $\{V_j\}, j\,\epsilon\,\mathbb{Z}$, with the scaling function φ, how do we find $2^n - 1$ filters $m_l(\xi)$ such that the $2^n \times 2^n$ matrix $\mathcal{M}(\xi)$ having rows $\{m_l(\xi + \frac{1}{2}\,\varepsilon\}, \varepsilon\,\epsilon\,\vartheta_n$, is unitary for a.e. ξ? It is clear from our discussion that once this is achieved, then the collection $\Psi = \{\psi^1, \psi^2, \ldots, \psi^{2^{n-1}}\}$ generates a wavelet basis for $L^2(\mathbb{R}^n)$. In this connection let us make the following observation. Since $\mathcal{M}(\xi)$ is unitary a.e., its "first column";

$$\begin{bmatrix} m_0(\xi) \\ m_1(\xi) \\ \cdots \\ m_{2^n-1}(\xi) \end{bmatrix}$$

is a vector of norm 1:

$$\sum_{l=0}^{2^{n-1}} m_l(\xi)\overline{m_l(\xi)} = 1.$$

Thus, $\sum_{l=0}^{2^{n-1}} \overline{m_l(\xi)}\,\widehat{\psi}^l(2\,\xi) = \sum_{l=0}^{2^{n-1}} |m_l(\xi)|^2\,\widehat{\varphi}(\xi) = \widehat{\varphi}(\xi)$. This shows that

$$\varphi\,\varepsilon\,\overline{\text{span}\{\psi^l_{-1,k} : l = 0, 1, 2, \ldots, 2^n - 1, k\,\epsilon\,\mathbb{Z}^n\}} \subset V_{-1} \oplus W_{-1}.$$

Since this last space equals V_0 which is shift invariant and is spanned by $\{\varphi(\cdot - k)\}, k\,\epsilon\,\mathbb{Z}^n$, it follows that the span of the functions $\psi^l_{-1,k}, 1 \leqslant l \leqslant 2^n - 1$, just considered, must be W_{-1}. We can conclude, therefore, that Ψ generates a wavelet basis.

It is natural to consider the problem of finding the filters $m_l, 1 \leqslant l \leqslant 2^n - 1$, given the filter m_0 associated with the scaling function m_0. We begin by looking for a filter $m_1(\xi)$ such that the vector $\{m_1(\xi + \frac{1}{2}\varepsilon)\}, \varepsilon\,\epsilon\,\vartheta_n$, has norm one and is orthogonal to $\{m_0(\xi + \frac{1}{2}\,\varepsilon)\}, \varepsilon\,\epsilon\,\vartheta_n$, for a.e. ξ. This can be done in several ways. Here is a simple construction: Let $\mathbf{1} = (1, 1, \ldots, 1)\epsilon\,\mathbb{R}^n$ and $\varepsilon_0 = (1, 0, \ldots, 0, 0)\epsilon\,\mathbb{R}^n$. If we define $m_1(\xi) = \overline{e^{i\,\xi\cdot\mathbf{1}}\,m_0(\xi + \frac{\varepsilon_0}{2})}$, then the two vectors $\mathcal{M}_l(\xi) = \{m_l(\xi + \frac{1}{2}\varepsilon)\}, \varepsilon \,\epsilon\, \vartheta_n, l = 0, 1$ are orthogonal to each other and of unit length a.e.. This provides us with one of the desired wavelet basis generators ψ^1 if we define it by the equality $\widehat{\psi}^1(2\xi) = m_1(\xi)\widehat{\varphi}(\xi)$. The search for $\psi^2, \ldots, \psi^{2^n-1}$ is more involved. Perhaps a few observations in the case $n = 2$ provide us with an insight into this situation. If m_0 is real-valued let $\mathcal{M}_0(\xi) = (m_0(\xi_1, \xi_2), m_0(\xi_1, \xi_2 + \frac{1}{2}), m_0(\xi_1 + \frac{1}{2}, \xi_2), m_0(\xi_1 + \frac{1}{2}, \xi_2 + \frac{1}{2}))$, and define the row vectors

$$\mathcal{M}_l(\xi) = \mathcal{M}_0(\xi) L_l,$$

$l = 1, 2, 3$, where

$$L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These provide us with the unitary matrix

$$U(\xi) = \begin{bmatrix} \mathcal{M}_0(\xi) \\ \mathcal{M}_1(\xi) \\ \mathcal{M}_2(\xi) \\ \mathcal{M}_3(\xi) \end{bmatrix} = \begin{bmatrix} m_0(\xi) \ldots & m_0(\xi + \frac{1}{2}(1,1)) \\ m_1(\xi) \ldots & m_1(\xi + \frac{1}{2}(1,1)) \\ m_2(\xi) \ldots & m_2(\xi + \frac{1}{2}(1,1)) \\ m_3(\xi) \ldots & m_3(\xi + \frac{1}{2}(1,1)) \end{bmatrix}$$

such that the first column gives us the filters determining the collection $\Psi = \{\psi_1, \psi_2, \psi_3\}$ that generates a wavelet basis associated with the MRA we are considering: $\widehat{\psi}^l(2\,\xi) = m_l(\xi)\widehat{\varphi}(\xi),\ l = 0, 1, 2, 3$. If the components of $\mathcal{M}_0(\xi)$ are complex-valued let

$$\begin{aligned} &(s_0(\xi)m_0(\xi),\ s_1(\xi)m_0(\xi + \frac{1}{2}(0,1)), \\ &\qquad s_2(\xi)m_0(\xi + \frac{1}{2}(1,0)),\ s_3(\xi)m_0(\xi + \frac{1}{2}(1,1,))) \\ &= \quad (|m_0(\xi)|, |m_0(\xi + \frac{1}{2}(0,1))|, |m_0(\xi + \frac{1}{2}(1,0))|, |m_0(\xi + \frac{1}{2}(1,1))|) \end{aligned}$$

and

$$S(\xi) = \begin{bmatrix} s_0(\xi) & 0 & 0 & 0 \\ 0 & s_1(\xi) & 0 & 0 \\ 0 & 0 & s_2(\xi) & 0 \\ 0 & 0 & 0 & s_3(\xi) \end{bmatrix},$$

where each $s_l(\xi)$ is unimodular, $l = 0, 1, 2, 3$. $S(\xi)$ is, then, a unitary matrix and, letting

$$\mathcal{M}_l(\xi) = \mathcal{M}_0(\xi) S(\xi) L_l S^*(\xi), \tag{4.14}$$

$l = 0, 1, 2, 3,$ we, again, obtain the desired filters by selecting the first components of these vectors. The matrices I, L_1, L_2, L_3 can be considered to represent the generators $1, i, j, k$ of the quaternions (observe that $L_l^2 = -I, l = 1, 2, 3,\ L_1 L_2 = L_3, L_2 L_3 = 1$ and $L_3 L_1 = L_2$). We can try to extend the idea of this construction by using the Cayley numbers and, for higher dimensions, the Clifford algebras. One encounters, however, some difficulties by following this path. Even in the two-dimensional case, the vectors defined by (4.14) may lack desired smoothness due to the discontinuities of the signum functions. Thus, even if the scaling function is compactly supported and has other desirable properties, we cannot expect the wavelets so obtained to be compactly supported.

One can obtain compactly supported complex-valued wavelets from an MRA in $\mathbb{R}^2$, however, by taking tensor products of 1-dimensional MRA's. The general case can be easily understood once we present the following two-dimensional case. Let θ be a scaling function in $L^2(\mathbb{R})$ that is compactly supported with an accompanying low pass filter s such that the wavelet ζ satisfying $\widehat{\zeta}(2\,\xi) = e^{2\,\pi\,i\,\xi}\overline{s(\xi)}\,\widehat{\theta}(\xi)$ is compactly supported (see chapter 2 of [HW] where the Daubechies wavelets are constructed; $s(\xi)$ in this case is a trigonometric polynomial). Then $\varphi(x, y) = \theta(x)\theta(y)$ is a scaling function for an MRA in $L^2(\mathbb{R}^n)$ and the polynomials $m_0(\xi) = m_0(\xi_1,\ \xi_2) = s(\xi_1)s(\xi_2),\ m_1(\xi) = s(\xi_1)e^{2\,\pi\,i\,\xi_2}\overline{s(\xi_2 + \frac{1}{2})}$, $m_2(\xi) = e^{2\,\pi\,i\,\xi_1}\,\overline{s(\xi_1 + \frac{1}{2})}s(\xi_2),\ m_3(\xi) = e^{2\,\pi\,i(\xi_1+\xi_2)}\,\overline{s(\xi_1 + \frac{1}{2})\,s(\xi_2 + \frac{1}{2})}$ will then provide us a unitary matrix $U(\xi)$, as in equality (4.14), from which we obtain the system of compactly supported wavelets $\Psi(x, y) = \{\psi^1(x, y), \psi^2(x, y), \psi^3(x, y)\}$ satisfying $\widehat{\psi^l}(2\,\xi) = m_l(\xi)\widehat{\varphi}(\xi), l = 1, 2, 3$. It is clear that this construction extends to n-dimensions and it gives us a compactly supported scaling function $\varphi(x) = \varphi(x_1, \ldots, x_n)$ that will produce a system $\Psi(x) = \{\psi^1(x), \ldots, \psi^{2n-1}(x)\}$ of compactly supported wavelets; furthermore, the 1-dimensional scaling functions whose product is φ can be different scaling functions of the variables $x_1, x_2, \ldots, x_n$.

Wavelets such as the ones we just described, as well as more general tensor products obtained by partitioning the variables $x_1, x_2, \ldots, x_n$ into m subsets, are sometimes called *separable*. Construction of wavelets having smooth filters is challenging even in the separable case, if we require that they not be tensor products of one dimensional functions (see [A] for an elegant, but not simple, such construction).

There exist wavelet bases for $L^2(\mathbb{R}^n)$ that are generated by single functions. They can be constructed directly by using the basic equations (A), (B), (C), and (D) (see [SoW]) or their existence can be established by employing operator theoretic methods (see [DL]). Clearly, these cannot be MRA wavelets. The wavelets obtained in [SoW] are MSF wavelets; that is, the absolute value of their Fourier transforms are characteristic functions of a set $W \subset \mathbb{R}^n$, a *wavelet* set. Such sets are "fractal" and enjoy various interesting properties that are described in [SoW].

It is natural to ask if there is a characterization of MRA wavelets. The answer is that there exists a simply stated condition that determines whether a function (or a collection of functions) is a wavelet obtained from an MRA. Let us first consider the one dimensional case and suppose ψ is an MRA wavelet; that is, $\widehat{\psi}(2\,\xi) = m_1(\xi)\widehat{\varphi}(\xi), \widehat{\varphi}(2\,\xi) = m_0(\xi)\widehat{\varphi}(\xi)$ and m_0, m_1 are 1-periodic functions satisfying, in particular,

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = 1,$$

for a.e. ξ (this follows from the 1-dimensional versions of the fact that $\mathcal{M}(\xi)$ is unitary). Consequently,

$$|\widehat{\psi}(2\xi)|^2 + |\widehat{\varphi}(2\xi)|^2 = \{|m(\xi)|^2 + |m_0(\xi)|^2\}|\widehat{\varphi}(\xi)|^2 = 1 \cdot |\widehat{\varphi}(\xi)|^2$$

a.e. Iterating this argument we obtain

$$|\widehat{\varphi}(\xi)|^2 = |\widehat{\varphi}(2^N\,\xi)|^2 + \sum_{j=1}^{N} |\widehat{\psi}(2^j\,\xi)|^2$$

a.e. for each positive integer N. It is clear that the limits

$$\lim_{N\to\infty} \sum_{j=1}^{N} |\widehat{\psi}(2^j\,\xi)|^2 \text{ and } \lim_{N\to\infty} |\widehat{\varphi}(2^N\,\xi)|^2$$

exist (observe that these sums are bounded and increasing) and the integrability of $|\widehat{\varphi}|^2$, together with Fatou's lemma, imply that the last limit is 0 a.e.. Thus,

$$|\widehat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\,\xi)|^2 \text{ a.e.} \tag{4.15}$$

Since $\{\varphi(\cdot - k) : k\,\epsilon\mathbb{Z}\}$ is an orthonormal system, we have, by Proposition I,

$$D_\psi(\xi) \equiv \sum_{j=1}^{\infty} \sum_{k\,\epsilon\,\mathbb{Z}} |\widehat{\psi}(2^j(\xi + k))|^2 = 1 \tag{4.16}$$

a.e.. The 1-periodic function D_ψ (clearly integrable on $[0, 1)$ whenever $\psi\,\epsilon\,L^2(\mathbb{R})$) is known as the *dimension function*. Equality (4.16) characterizes the 1-dimensional MRA wavelets:

Theorem. *Suppose $\psi\,\epsilon\,L^1(\mathbb{R})$ is a wavelet. Then ψ is an* MRA *wavelet if and only if* $D_\psi(\xi) = 1$ a.e..

See chapter seven of [HW] for a discussion and appropriate credits for this result.

The dimension function D_ψ can be defined by equality (4.16) for any $\psi\,\epsilon\,L^2(\mathbb{R}^n)$ (the only change is that k now ranges throughout $\mathbb{Z}^n$). Essentially the same argument we have just given shows that if

$$\Psi = \{\psi^1, \psi^2, \dots \psi^L\}, L = 2^n - 1$$

is an MRA wavelet system, then

$$D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \,\epsilon\, \mathbb{Z}^n} |\widehat{\psi}^l(2^j(\xi + k))|^2 = 1 \tag{4.17}$$

a.e.. We also observe that this equality can only hold if $L = 2^n - 1$. Indeed, if (4.17) is true, then

$$1 = \int_{\mathbb{T}^n} D_\Psi(\xi) d\,\xi = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \,\epsilon\, \mathbb{Z}^n} \int_{\mathbb{T}^n} |\widehat{\psi}(2^j(\xi + k))|^2 d\,\xi$$

$$= \sum_{l=1}^{L} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\widehat{\psi}(2^j\,\xi)|^2 d\,\xi = \sum_{l=1}^{L} \sum_{j=1}^{\infty} 2^{-jn} ||\psi^l||_2^2 = L\left(\frac{1}{2^n - 1}\right).$$

5. The connectivity of wavelets.

Let us return to the "classical" 1-dimensional wavelets. We shall discuss the connectivity of this class. We begin by showing that the set of MRA wavelets is an arcwise connected set:

THEOREM (5.1). *If ψ_0 and ψ_1 are two MRA wavelets, then there exists a continuous map $A : [0,1] \longrightarrow L^2(\mathbb{R})$ such that $A(0) = \psi_0, A(1) = \psi_1$ and $A(t)$ is an MRA wavelet for all $t \,\epsilon\, [0,1]$.*

This result is due to Xingde Dai and Rufeng Liang . Their proof is presented in [Wu] (they are members of the Wutam Consortium). We shall present the basic ideas of their argument. In order to do so we will use some of the notions introduced in [Wu]. We begin by introducing three "multipliers" that play important roles in the theory of wavelets:

Definition. A measurable function ν on $\mathbb{R}$ is a *wavelet multiplier* if and only if $(\nu\,\widehat{\psi})^\vee$ is an o.n. wavelet whenever ψ is an o.n. wavelet. ν is a *scaling function multiplier* if and only if $(\nu\,\widehat{\varphi})^\vee$ is a scaling function whenever φ is a scaling function. A measurable function μ is a *low pass filter multiplier* if and only if $\mu\, m$ is a low pass filter whenever m is a low pass filter.

If $(\nu\,\widehat{\psi})^\vee$ is an MRA wavelet whenever ψ is an MRA wavelet we say that ν is an *MRA wavelet multiplier*.

These multipliers have been completely characterized in [Wu]:

THEOREM (5.2). *ν is a wavelet, MRA wavelet or scaling function multiplier if and only if it is unimodular and $\nu(2\,\xi)/\nu(\xi)$ is a.e.equal to a 1-periodic function. μ is a low pass filter multiplier if and only if it is a unimodular function that is equal a.e. to a 1-periodic function.*

(The term "unimodular function" means that the function in question has absolute value 1 a.e.).

A few observations are in order. If ψ_0 is an MRA wavelet then the other wavelets belonging to the same MRA are precisely those functions ψ such that $\widehat{\psi} = \nu\,\widehat{\psi}_0$ where ν is unimodular and a.e. equal to a 1-periodic function. If ν is a wavelet multiplier, $\psi = (\nu\,\widehat{\psi}_0)^\vee$ may very well belong to a different MRA (by the last theorem in section 4 and Theorem (5.2), however, we do know that such a ψ is an MRA wavelet). We shall be interested in the classes

$$\mathcal{W}_{\psi_0} = \{\text{all wavelets } \psi : |\widehat{\psi}(\xi)| = |\widehat{\psi}_0(\xi)| \text{ a.e.}\}$$

whenever ψ_0 is a wavelet. It follows from the above observations that if ψ_0 is an MRA wavelet then each $\psi \,\epsilon\, \mathcal{W}_{\psi_0}$ is an MRA wavelet and $\mathcal{W}_{\psi_0}$ contains all the wavelets generated by the MRA that produced ψ_0. Furthermore, if $\psi = (\nu\,\widehat{\psi}_0)^\vee$, where ν is a wavelet multiplier, then $\psi \,\epsilon\, \mathcal{W}_{\psi_0}$. Thus, if, for a wavelet ψ_0, we let

$$\mathcal{M}_{\psi_0} = \{\psi = (\nu\,\widehat{\psi}_0)^\vee : \nu \text{ a wavelet multiplier}\},$$

then

$$\mathcal{M}_{\psi_0} \subset \mathcal{W}_{\psi_0} \tag{5.3}$$

We shall consider a third class of wavelets generated by an MRA wavelet ψ_0. In order to introduce this class it is helpful to use the following characterization of the scaling functions (see chapter 7 of [HW]):

THEOREM (5.4). *A function $\varphi \,\epsilon\, L^2(\mathbb{R})$ is a scaling function for an MRA if and only if*

1. $\sum_{k\,\epsilon\,\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2 = 1$ *a.e.*;
2. $\lim_{j\to\infty} |\widehat{\varphi}(2^{-j}\,\xi)| = 1$ a.e.;
3. *there exists a 1-periodic* $m \,\epsilon\, L^2([-1/2, 1/2))$ *such that*
$$\widehat{\varphi}(2\,\xi) = m(\xi)\widehat{\varphi}(\xi).$$

An immediate consequence of this theorem is that $|\widehat{\varphi}|$ is the Fourier transform of a scaling function whenever φ is a scaling function. The function m is, of course, the low pass filter associated with the scaling function φ via the equality in 3 of Theorem (5.4); furthemore, m is uniquely determined by φ and satisfies the Smith-Barnwell equality

$$|m(\xi)|^2 + |m(\xi + \frac{1}{2})|^2 = 1 \tag{5.5}$$

for a.e. ξ. The most general wavelet belonging to the MRA generated by φ satisfies

$$\widehat{\psi}(2\,\xi) = e^{2\,\pi\,i\,\xi}\, s(2\,\xi)\overline{m(\xi + \frac{1}{2})}\widehat{\varphi}(\xi) \tag{5.6}$$

where s is a unimodular 1-periodic function. An immediate consequence of (5.5) and (5.6) is that if φ^1 and φ^2 are two scaling functions such that $|\widehat{\varphi}^1(\xi)| = |\widehat{\varphi}^2(\xi)|$ a.e. and ψ^1, ψ^2 are MRA wavelets associated with φ^1 and φ^2, then $|\widehat{\psi}^1(\xi)| = |\widehat{\psi}^2(\xi)|$ a.e. It follows that the class $\mathcal{S}_{\psi_0}$ of all MRA wavelets associated with a scaling function φ satisfying $|\widehat{\varphi}(\xi)| = |\widehat{\varphi}_0(\xi)|$ a.e. (ψ_0 being an MRA wavelet associated with φ_0) is a subset of $\mathcal{W}_{\psi_0}$. On the other hand, equality (4.15) shows that $\mathcal{W}_{\psi_0} \subset \mathcal{S}_{\psi_0}$ Thus,

$$\mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0} \tag{5.7}$$

whenever ψ_0 is an MRA wavelet. In fact we have

THEOREM (5.8). *If ψ is an MRA wavelet, then*

$$\mathcal{M}_\psi = \mathcal{W}_\psi = \mathcal{S}_\psi.$$

This result is proved in [Wu].

The sets $\mathcal{M}_\psi$ and $\mathcal{W}_\psi$ can be defined for any wavelet ψ, not necessarily MRA. In view of Theorem (5.8), it is natural to ask if the equality $\mathcal{M}_\psi = \mathcal{W}_\psi$ is true for *all* wavelets ψ. The answer is “No;” Q. Gu constructed a clever counterexample.

Let us describe the two basic ideas of the proof of this theorem. First one shows that each of the classes $\mathcal{W}_\psi, \psi$ an MRA wavelet, is connected. Second, one shows that each MRA wavelet ψ_1 can be connected to the Shannon wavelet ψ_0. As we shall see, this part of the argument is facilitated by the fact that we can choose ψ_1, by Theorem (5.8), so that its associated scaling function φ_1 satisfies $\widehat{\varphi}_1(\xi) \geqslant 0$ a.e. (since $\mathcal{W}_{\psi_1} = \mathcal{S}_{\psi_1}$ this clearly can be done).

For the first part of this proof, which establishes that $\mathcal{W}_\psi$ is connected, one chooses $\psi_1 \,\epsilon\, \mathcal{W}_\psi = M_\psi$ (by Theorem (5.8)) so that $\widehat{\psi}_1 = \nu\,\widehat{\psi}$ for an appropriate wavelet multiplier. Since ν is unimodular (Theorem (5.2)) we can write $\nu(\xi) = e^{i\,\lambda(\xi)}$; $\lambda(\xi)$, however, is not unique, but it is easy to construct an appropriate λ so that $\nu_t(\xi) = e^{it\,\lambda(\xi)}$ is a wavelet multiplier ($\nu_t(2\,\xi)\overline{\nu_t(\xi)}$ is 1-periodic) for $t\,\epsilon\,[0,1]$. We then obtain a continuous map $\theta : t \longrightarrow \psi_t \equiv (\nu_t\,\widehat{\psi})^\vee$ on $[0,1]$ such that for $t = 0$ we have $\theta(0) = \psi_0 = \psi$ and, for $t = 1$, we have $\theta(1) = \psi_1$.

The second part of the proof is somewhat more involved. We present the basic features of the argument. As explained above, we can assume that we have a wavelet ψ_1 constructed from a scaling function φ_1 such that $\widehat{\varphi}_1(\xi) \geqslant 0$ a.e.. It suffices to connect ψ_1 to the Shannon wavelet ψ_0 whose scaling function φ_0 satisfies $\widehat{\varphi}_0(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$. It follows that the corresponding filters, m_0 and m_1, are non-negative. It is tempting to define

$$m_t(\xi) = \sqrt{(1-t)m_0(\xi)^2 + tm_1(\xi)^2} \tag{5.9}$$

for $t \epsilon [0,1]$ (observe that the Smith-Barnwell equality, $m_t(\xi)^2 + m_t(\xi + \frac{1}{2})^2 = 1$ is true). This provides us with a continuous path of low pass filters. The corresponding scaling functions satisfying

$$\widehat{\varphi}_t(\xi) = \Pi_{j=1}^{\infty} m_t(2^{-j}\,\xi)$$

form a continuous path of scaling functions. The desired path of wavelets is, then, obtained by letting

$$\widehat{\psi}_t(2\,\xi) = e^{2\,\pi\,i\,\xi}\overline{m_t(\xi + \pi)}\widehat{\varphi}_t(\xi)$$

for $t \epsilon [0,1]$. This scheme "almost works". The main modification needed is that the intermediate filters be obtained by an equality that is technically more complicated than (5.9) (see [Wu] page 588).

The consideration of the connectivity of wavelets is very natural. The "first wavelets" (besides the Haar and Shannon wavelets), constructed in the early eighties by Lemarié and Meyer, are, in a real sense, obtained by a continuous "smoothing" (on the Fourier transform side) of the Shannon wavelet. A general result on connectivity was obtained in [BDW]. The authors in this work concerned themselves with wavelets produced by very smooth filters. The paths obtained were continuous with respect to a topology that is considerably stronger than that produced by the $L^2(\mathbb{R})$-norm. There is a topological impediment that prevents one from connecting two wavelets in general when this stronger topology is used. For example, if ψ_0 is the Haar wavelet and $\psi_1(x) = \psi_0(x-1)$ is its translate by 1, then $\widehat{\psi}_1(\xi) = s(\xi)\widehat{\psi}_0(\xi) = e^{-i\,\xi}\widehat{\psi}_0(\xi)$. The fact that the function that is identically 1 and $s(\xi)$ are not in the same homotopy class prevents the existence of a path joining these two wavelets that is continuous in the topology used in [BDW]. These questions, as well as extensions of the results in this last work, are discussed by G. Garrigós in [Ga].

An interesting connectivity result was obtained by D. Speegle [SP] who showed that the collection of MSF wavelets is connected. Let us say a few words to put this result in some perspective. One of the first examples of a wavelet that is not an MRA wavelet was obtained by J-L Journée. His wavelet is an MSF wavelet (see [HW], page 64). It follows that the union of the MRA and MSF wavelets is an arcwise connected subset of the surface of the unit sphere in $L^2(\mathbb{R})$. Auscher [A] and Lemarié [Le] have, independently, shown that if one makes rather mild assumptions about the Fourier transform of a wavelet (continuity and a decrease at ∞), the wavelet arises from an MRA. We see therefore, that "most" wavelets are either MRA or MSF wavelets. It is not unreasonable to conjecture, therefore, that the collection of all wavelets in $L^2(\mathbb{R})$ is connected. The answer to this conjecture, however, is not yet known.

If we consider the question of connectivity and, more generally, multipliers, in connection with higher dimensional wavelets produced by more general dilations we find various factors that alter the form of the results we seek. We have seen, for example, that the MRA wavelets in higher dimensions require more than one generator. This, however, depends on the dilation that is used. This is not the case for those wavelets on $\mathbb{R}^2$ obtained by the translations by the points of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and the dilations obtained from the integral powers of the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

That is, if $\psi \,\epsilon\, L^2(\mathbb{R}^2)$, the system $\{\psi_{jk}\}, j \,\epsilon\, \mathbb{Z}, k \,\epsilon\, \mathbb{Z}^2$ is defined by

(5.10) $$\psi_{jk}(x) = (\det M)^{j/2}\, \psi(M^j x - k) = 2^{j/2}\, \psi(M^j x - k)$$

for $x \,\epsilon\, \mathbb{R}^2$ ($M^j x$ denoting the vector obtained by multplying the matrix M^j by the column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \epsilon\, \mathbb{R}^2)$$

A function ψ such that $\{\psi_{jk}\}, j \,\epsilon\, \mathbb{Z}, k \,\epsilon\, \mathbb{Z}^2$, is an orthonormal basis for $L^2(\mathbb{R}^2)$ is called a *Quincunx wavelet*. There exist MRA wavelets, ψ, of this type producing orthonormal bases of the form $\{\psi_{jk}\}$, just as in the one-dimensional case. That is, even though the dimension of the underlying space, $\mathbb{R}^2$, is 2, we do not need $3\,(= 2^2 - 1)$ generators to obtain a "wavelet basis" for $L^2(\mathbb{R}^2)$ as is the case described in section 4.

The quincunx wavelets share many other properties with the "classical" wavelets on $\mathbb{R}$ obtained by integral translations and dyadic dilations. In particular, the Wutam program, for the quincunx wavelets, is just like the one we just presented.

Let $\psi \,\epsilon\, L^2(\mathbb{R}^2)$ and $\{\psi_{jk}\}$ be the system defined by equality (5.10). Then, letting M_1 be the transpose of M, we have

(5.11) $$\widehat{\psi}_{jk}(\xi) = 2^{-j/2}\, \widehat{\psi}(M_1^{-j}\, \xi) e^{-2\,\pi\, ik\cdot M_1^{-j}\, \xi}$$

for $\xi \,\epsilon\, \mathbb{R}^2, j \,\epsilon\, \mathbb{R}$ and $k \,\epsilon\, \mathbb{Z}^2$. Until further notice, we shall use the term "wavelet" for an orthonormal quincunx wavelet (that is, a function $\psi \,\epsilon\, L^2(\mathbb{R}^2)$ such that the system defined by (5.10) is an o.n. basis for $L^2(\mathbb{R}^2)$). The characterization of wavelets obtained by general dilations, Theorem (2.3), reduced to the situation we are now studying, becomes

PROPOSITION (5.12).. *Let $\psi \,\epsilon\, L^2(\mathbb{R}^2)$ such that $||\psi||_2 \geqslant 1$. Then ψ is an o.n. wavelet if and only if*

(I) $$\sum_{j\,\epsilon\,\mathbb{Z}} |\widehat{\psi}(M_1^j\, \xi)|^2 = 1 \text{ for a.e. } \xi \,\epsilon\, \mathbb{R}^2,$$

(II) $$\sum_{j \geqslant 0} \widehat{\psi}(M_1^j\, \xi)\overline{\widehat{\psi}(M_1^j(\xi + q))} = 0 \text{ for a.e. } \xi \,\epsilon\, \mathbb{R}^2$$

whenever $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \epsilon\, \mathbb{Z}^2$ *with* q_1, q_2 *having different parity*.

Let $\mathbb{T} = \mathbb{T}^2 = \{\xi = (\xi_1, \xi_2) : -\frac{1}{2} \leqslant \xi_j < \frac{1}{2}, j = 1, 2\}$ and

$$S = (M\mathbb{T}) \backslash \mathbb{T}$$

($M\,\mathbb{T}$ is the square with vertices (0,1), (-1,0), (0,-1) and (1,0)). Letting ψ be defined by having $\widehat{\psi}(\xi) = \mathcal{X}_s(\xi)$, it is easy to check that ψ is an orthonormal wavelet that is also an MRA wavelet. A scaling function ψ for the MRA is obtained by letting $\widehat{\varphi}(\xi) = \mathcal{X}_{\mathbb{T}}(\xi)$ and its associated low pass filter m is the characteristic function of $\mathcal{X}_M^{-1}\,\mathbb{T}$, restricted to $\mathbb{T}$, and then extended to be 1-periodic in each variable. In fact, this provides us with an example of an MRA, MSF wavelet in the quincunx case (the analogue of the Shannon wavelet).

The notion of multiplier (wavelet, scaling function or filter multiplier) as well as the definition of the sets W_ψ, $\mathcal{M}_\psi$ and S_ψ extends in an obvious way, and so does Theorem (5.8). In fact, it is not hard to obtain the version of Theorem (5.1) that asserts that the *MRA quincunx wavelets are path-connected*. The details of this "Wutam program" adapted to this situation were presented in a Ph.D. qualifying oral exam by one of our students, L. Zhang.

6. Summary and Bibliographical comments.

As we stated at the beginning, one of our motivations was to present the "Mathematical Theory of Wavelets." Obviously, we cannot do this in any exhaustive way in this rather short article. Our aim was to describe some of the beauty of this theory and, if possible, elaborate topics that are new. We hope that this write-up does have some of these properties.

We have not considered many important uses of wavelet theory in mathematics. For example, their application to the construction and deriving properties of a large collection of important function spaces (the Besov and Triebel-Lizorkim spaces) is not a topic we discussed. Chapters 5 and 6 in [HW] and [FJW] deal with this topic.

We described various kinds of wavelets, scaling functions, filters and we gave characterizations of almost all these function with the exception of low pass filters. One of us was involved in the characterizations of *all* low pass filters in 1-dimension and for dyadic dilations [PSW]. The characterization of the dual systems (Φ, Ψ) announced by Theorem (2.3) really pertains to the case where the dilation matrix A has integer entries and the translations are obtained from the lattice $\mathbb{Z}^n$ (in terms of the notation used in section 2, the matrix $P^{-1}\,A\,P$ must have integer entries and this similarity reduces the problem to the lattice $\mathbb{Z}^n$). There are, however, wavelet systems that involve more general dilations and translations; in [CCMW] the problem of the characterization of such systems is solved.

Though we concentrated our attention to wavelets, we indicated that other systems, obtained by applying, say, modulations and translations to a fixed function, are also of interest. For example, systems of the form

$$g_{mn}(x) = e^{2\pi imx} g(x-n), \tag{6.1}$$

$m, n \,\epsilon\, \mathbb{Z}$, the *Gabor systems*, have been studied extensively. A very general context that produces the "continuous" versions of such systems (as well as the continuous wavelets) can be described by the collection of $(n+2)\times(n+2)$ matrices of the form

$$g = \begin{pmatrix} 1 & b & z \\ 0 & A & c \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.2}$$

where b is an n-dimensional row vector, c an n-dimensional column vector, $A \,\epsilon\, GL(n, \mathbb{R})$ and $z \,\epsilon\, \mathbb{R}$. If we let the matrix (6.2) act on the column vector (on the right)

$$\begin{pmatrix} u \\ v \\ y \end{pmatrix},$$

where $u, y \,\epsilon\, \mathbb{R}$ and $v \,\epsilon\, \mathbb{R}^n$, we obtain the action

$$\begin{pmatrix} u \\ v \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} u + b\cdot v + zy \\ Av + cy \\ y \end{pmatrix}$$

This induces the following mapping

$$(T_g\,\psi)(u, v, y) = |\det A|^{-\frac{1}{2}} e^{2\pi i(u+b\cdot v+zy)} \psi(Av + cy)$$

defined on a function $\psi(v), v \,\epsilon\, \mathbb{R}^n$. If $u = 0, b = 0$ and $y = 1$, we obtain the map (2.1) (strictly speaking, we should use g^{-1} instead of g; the variable z is not important and its main function is to make sure that the matrices (6.2) form a group). If $A = I, y = 1, u = 0$ we obtain the "continuous Gabor system," which is also referred to as the *Weyl-Heisenberg* transform. It is natural to consider the various themes treated in the previous questions in this general setting. For example, we can try to find admissibility conditions, such as Theorem (2.1), when A belongs to a subgroup of $GL(n, \mathbb{R})$ and $b, c \,\epsilon\, \mathbb{R}^n$. In the Weyl-Heisenberg case this is considered in [LP]; see, also, the discussion in [DGM] that is relevant to this case. The authors of [LWWW] are engaged in an investigation of these problems in the general case. Particularly challenging is the question of the discretizations of the continuous transforms. The Gabor system

$$\psi_{mn}(x) = e^{2\pi imb\cdot x} \psi(x + nc)$$

is a particular discretization of this type that is well known; various conditions guaranteeing the orthonormality and frame properties of these functions have been considered by many authors (see, in particular, [Cz]).

We have included some items in the bibliography that are not referred to directly. In general these pertain to presentations of continuous wavelets and their discretizations that should be compared with our presentation; there are, also, certain aspects of what we describe here that are, formally, quite similar to work done in representations theory. We suggest that the reader examine [BGZ], [BT], [Car], [DM], [GM], [GMP_1], [GMP_2], [GP], [K], [LP], [M], and [ST] for the work done that is related to section 2.

One last word about the "characterization" equalities we talked about. Clearly it is a good thing to find out descriptions of the class of all functions having certain properties (wavelets, scaling functions, tight frame wavelets, low pass filters, etc.) We claimed, at the end of section 1, that the equalities we are considering have been very useful for many constructions. The book [HW] presents many examples. We cite [BGRW] as another example that attracted some attention.

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